18/03

Kinematics

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DIMES

position and trajectory

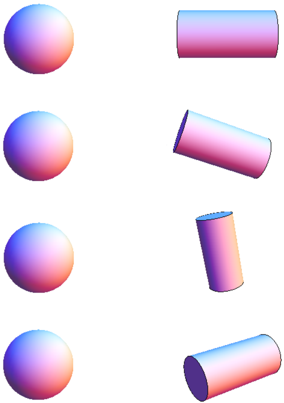
Reference system - point particle

- The concept of motion is linked to the definition of **reference system**
- **Reference system** is a set of bodies (whose relative positions do not vary) with respect to which to measure motion.
- **When we talk about the motion of an object, we must first specify the reference system**.
- If the size of the object is negligible, this can be represented as a point in space that maintains the physical properties (mass, charge, etc) of the original object: **point particle**.

Point particle

It is an abstraction that maintains the physical properties (mass, charge, etc) of the original object.

A point particle can only translate, if instead an object can not be schematized as a point particle (**extended body**) the description of the motion is more complicated, adding the rotation motion.

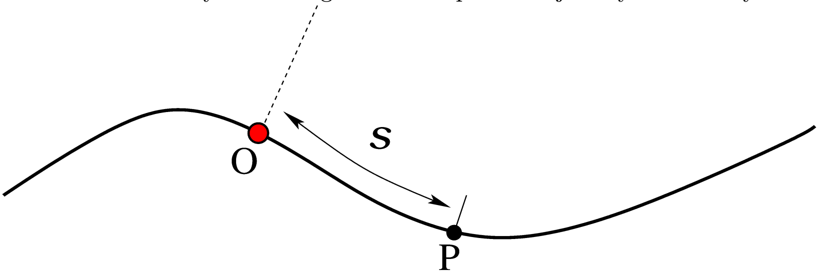


Trajectory $s = s(x, y)$ and motion law $s = s(t)$

- If a point moves (with respect to a prefixed reference system) it will occupy different positions in space as time changes.
- **trajectory**: set of positions occupied by the point during motion.
- Since each position is a point in space, the trajectory is a set of points, ie a **curve**
- From the knowledge of the trajectory, you can specify the **motion law**, that is how the position of the point varies with time.

Trajectory $s = s(x, y)$ and motion law $s = s(t)$

In the description of the motions another physical quantity is also important, the time, which allows us to have a description of the motion, not only geometric, and to introduce the concept of “motion that develops over time”. The geometric character of the motion and its dependence on time are introduced into Physics through the concepts of trajectory and hourly law.



Definition of the hourly law starting from the trajectory. By setting a O origin, we measure the length from the trajectory portion s that the point runs through moment by moment, that is, as a function of time.

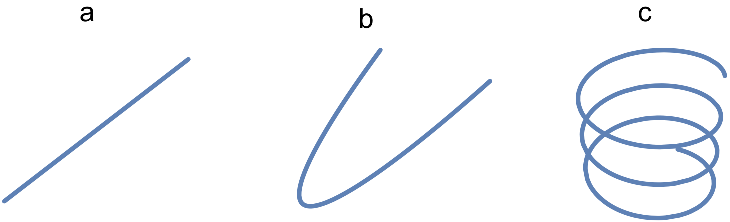
Dimension

number of coordinates necessary to uniquely identify the position of the object and its displacement.

- **unidimensional motion** is a motion identified by a single coordinate (along a straight line).
- **bidimensional**: two coordinates (on a plane)
- **three-dimensional**: three coordinates (in space).

The trajectory will be a **vector** of corresponding size

Examples



Medicine and Biology

- Position measurements, and trajectories are becoming increasingly important also in Medicine and Biology.
- Modern diagnostic instrumentation allows the visualization and localization of "points" (pixels, voxels) within the human body with a high degree of spatial resolution.
- For example, techniques such as radioisotope-based imaging, ultrasound, TAC (Computerized Axial Tomography) and Nuclear Magnetic Resonance allow visualization of organs and tissues with spatial resolutions of the order of mm or even less.

Medicine and Biology

Many biomedical signals (functional NMR, ECG, ecodoppler to name a few) are a function of time, and their analysis is useful for the discrimination between pathological and non-pathological states:

- an ECG trace, which shows the electrical activity of the heart as a function of time, allows to characterize any anomalies such as arrhythmias and fibrillations.
- functional NMR allows to observe changes in brain activity over time
- sequences of CT images during radiotherapy allow to monitor the progression of the tumor (size and shape)

Medicine and Biology

We can generalize as a trajectory the variation of a parameter x measured as a function of time.
Examples: temperature, heart rate, the level of expression of a protein, the rate of cell proliferation, perhaps following system perturbations (pathology, surgery, therapy).

$$x = x(t)$$

velocity and acceleration

Velocity and Acceleration

- derived quantities in Mechanics, Physics and Sciences in general.
- Acceleration, which in the common language is intended as an increase in speed, is defined starting from the velocity vector
- acceleration occurs even when a car slows down or curves (since in all these cases the velocity vector varies over time) and therefore the brakes and the steering wheel are accelerators in the same way.

Scalar velocity

We call *mean scalar velocity* in the time interval $\Delta t = t_2 - t_1$, the amount:

$$v_{sm} = \frac{\Delta s}{\Delta t} = \frac{s\left(t_2\right) - s\left(t_1\right)}{t_2 - t_1}$$

Dimensionally $v_{sm} = \frac{L}{T}$ thus velocity is expressed in units of length on unit of time $\left(\frac{m}{s}\right)$.

Vector velocity

If we take into account the direction of motion we have the concept of *vector velocity*:

$$\vec{v}_{vm} = \frac{\Delta \vec{s}}{\Delta t} = \frac{\vec{s}(t_2) - \vec{s}(t_1)}{t_2 - t_1}$$

the average vector velocity module coincides with the mean scalar velocity only in the case of 1-d motion, otherwise they may be different.

Instantaneous speed

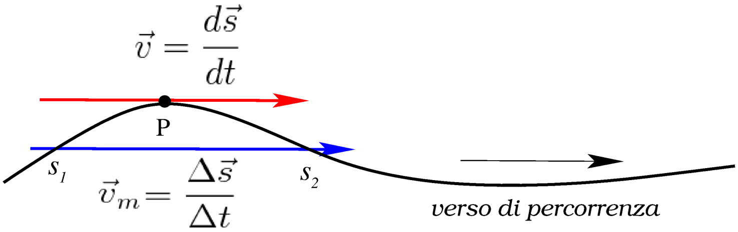
- The *instantaneous scalar velocity* is defined as the mean scalar velocity in the limit where Δt tends to zero.
- Conceptually, we can think of increasing the accuracy in the description of a motion by calculating the average velocities on ever smaller Δs intervals, until the average velocity coincides with the instantaneous velocity.
- This procedure is expressed mathematically by passing to the limit:

$$v = \lim_{t_2 \to t_1} \frac{s(t_2) - s(t_1)}{t_2 - t_1} = \lim_{\Delta t \to 0} \frac{\Delta s}{\Delta t} = \left(\frac{ds}{dt}\right)_{t=t_1} = v(t_1)$$

Instantaneous speed

- The direction of $\vec{v}$ is determined by the direction of travel of the trajectory.
- The amount ds/dt is called scalar velocity. Said $\vec{u}_{tg}$ the tangent to the trajectory, you can write: $\vec{v} = \frac{ds}{dt}\vec{u}_{tg}$.
- The velocity module measures the rapidity of movement of the material point along the trajectory and, in the S.I., is measured in m/s.
- In general, the velocity vector varies over time in module, direction and verse, ie: $\vec{v} = \vec{v}\,(t)$.

Average and instantaneous speed

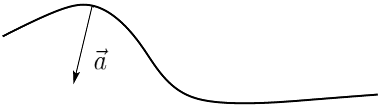


- The instantaneous velocity of a material point is a vector tangent to the trajectory at the point where it is instantaneously and in form $|\vec{v}| = \left|\frac{d\vec{s}}{dt}\right|$.
- The average velocity is a vector that is secant to the trajectory and passing through the points $s_1 = s(t_1)$ e $s_2 = s(t_2)$

Acceleration

- When the speed of a body varies over time, it is called accelerated motion.
- Also acceleration can be defined as an instantaneous average.
- Acceleration, in straight motions, has the same direction as the trajectory of the material point.
- In non-rectilinear trajectories, the acceleration $\vec{a}$ is a vector that lies in the trajectory plane, and is directed inside the concavity formed by the trajectory

Average and instantaneous



$$\vec{a}_m = \frac{\vec{v_1} - \vec{v_0}}{t_1 - t_0} = \frac{\Delta \vec{v}}{\Delta t}$$

$$\vec{a} = \lim_{t_1 \to t_0} \frac{\vec{v_1} - \vec{v_0}}{t_1 - t_0} = \lim_{\Delta t \to 0} \frac{\Delta \vec{v}}{\Delta t}$$

Acceleration is a vector directed towards the concavity of the trajectory

- Straight motion ⇒ constant direction
- If it is a straight motion, the average and instantaneous acceleration are directed along the trajectory:

$$a = \frac{dv}{dt} = \frac{d\left(\frac{ds}{dt}\right)}{dt} = \frac{d^2 s}{dt^2}$$

Fundamental 1-d motions

Fundamental 1-d motions

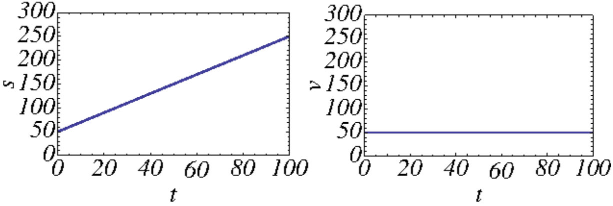
Motions in several dimensions can be seen

- as exact combinations of these one-dimensional motions (e.g. parabolic or circular motion)
- approximated by these motions (for example, any trajectory traveled with approximately constant velocity it can be seen as a succession of rectilinear motions and uniform circular motions).

Uniform rectilinear motion

- Point particle moving at constant speed along a rectilinear trajectory
- Speed has same direction than trajectory, and constant modulus $|ds/dt|$.
- Constant speed, no accelaration ($a(t) = 0$), linear motion law:

$$s\left(t\right) = s_0 + v_0 t$$



Examples

Uniform rectilinear motion when the resultant of the forces applied on the point P is zero.
For example, when the forces that oppose the motion have intensity that depends on the speed (friction forces)

- the motion of a body in free fall in the air
- the motion of particles, falling free or by centrifugation, in a viscous medium (sedimentation)
- the motion of electrically charged particles in a conductor (current) or in viscous solutions (electrophoresis)

In general, the presence of friction causes the application of a constant "propulsive" force to produce a uniform rectilinear motion rather than uniformly accelerated (described in the following section).

Examples

A plane wave in a uniform medium In general, although it can change the direction of motion (eg spherical wave) the propagation speed of a wave is constant.

Uniformly accelerated rectilinear motion

Constant acceleration $a(t) = a_0$: average and instantaneous acceleration are equal.
Rectilinear motion: constant speed in the direction (which coincides with the trajectory) but varies in the modulus:

$$v(t) = v_0 + at$$

$v(t)$ varies linearly with time and the speed of variation is proportional to acceleration. The movement depends in a quadratic way on the time:

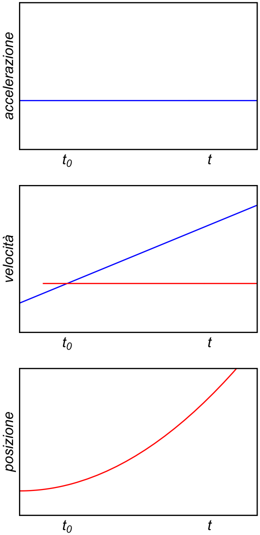
$$s(t) = s_0 + v_0 t + \frac{1}{2}at^2$$

velocity based on the travel distance

From the equations for $s(t)$ and $v(t)$ we obtain (obtaining t from the second relation and replacing it in the first) the final velocity according to the space traveled:

$$v(t) = \sqrt{v_0^2 + 2a(s(t) - s_0)}$$

Uniformly accelerated rectilinear motion



The uniformly accelerated motion is characterized by:

- (a) constant acceleration;
- (b) speed that varies linearly over time
- (c) position that varies quadratically with time

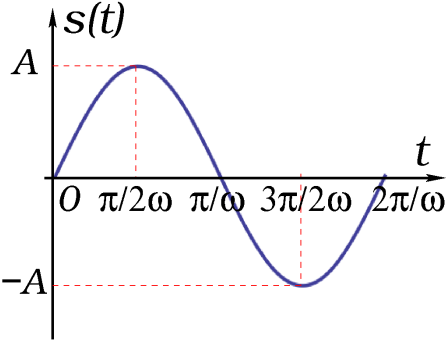
Harmonic motion

It is the simplest motion describing a point that oscillates around a position of equilibrium, which is obtained when the force applied to the point is proportional to the distance from the point of equilibrium.

$$s\left(t\right) = A\,sin\left(\omega \cdot t + \varphi\right)$$

- A: maximum distance that the point reaches with respect to the equilibrium position (amplitude of the oscillation),
- ω: number of point oscillations in 2π seconds (pulsation)
- φ: starting position (angle) of the oscillation at t_0 (initial phase).

Motion law



- Frequency ν: number of complete oscillations in 1 second: $\omega = 2\ \pi\ \nu$.
- Period T: time needed to complete a complete oscillation; $T = 1/\nu$

$$s_0 = A \sin(\phi)$$

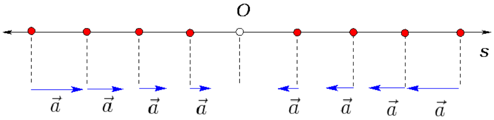
$$\sin(\phi) = \frac{s_0}{A}$$

The sine of the initial phase measures the fraction of amplitude at which the point is located at $t = 0$.

Acceleration

$$v(t) = \frac{ds\,(t)}{dt} \quad = \quad A\omega\cos(\omega t + \varphi)$$
$$a(t) = \frac{d^2s\,(t)}{dt^2} \quad = \quad -A\omega^2\sin(\omega t + \varphi) = -\omega^2 s$$

The acceleration is proportional to the shift s and directed in the opposite direction



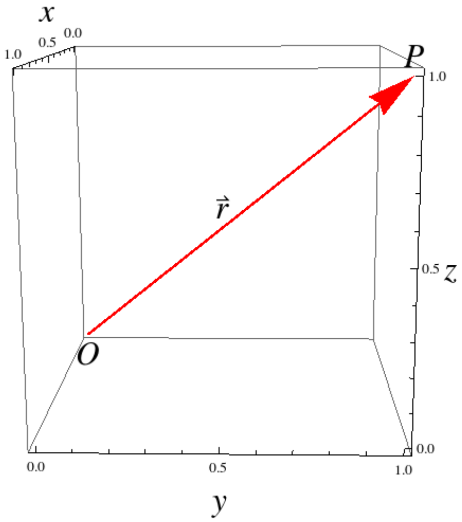
The harmonic motion is the first approximation of the effect of a perturbation on a body when it lies in a stable equilibrium position (corresponding to a minimum of its potential energy).

Multi-dimensional kinematics

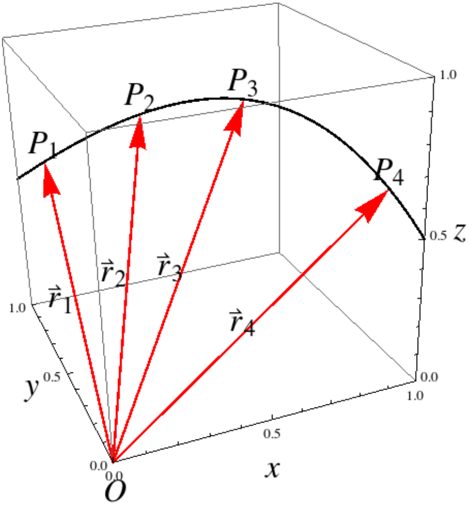
Multi-dimensional kinematics

- Everything we have seen for 1-d motions can be generalized to two and three dimensions.
- The position in space: 3-d vector (with three components along the axes x, y and z) and also speed and acceleration.
- The position vector is indicated with $\vec{r}$, and is used both to identify the position of a generic P point in three-dimensional space and to indicate the displacement of the P point on a three-dimensional trajectory.
- $\vec{r}$ has three components that are given by its projection on the x, y and z axes that are normally indicated with r_x, r_y, r_z.

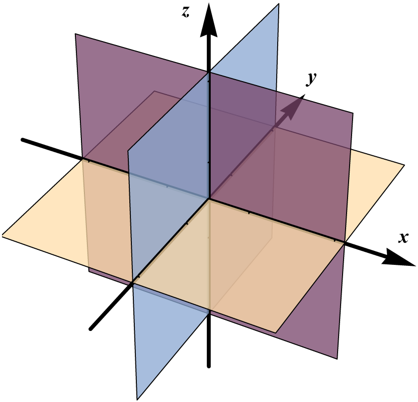
Position vector

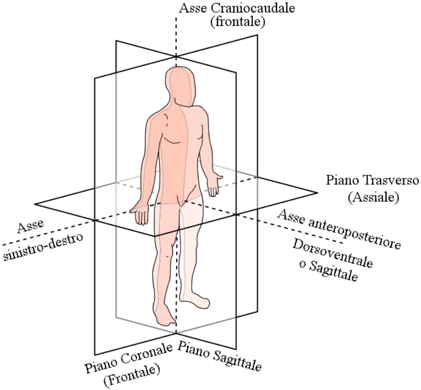


Position vector during motion

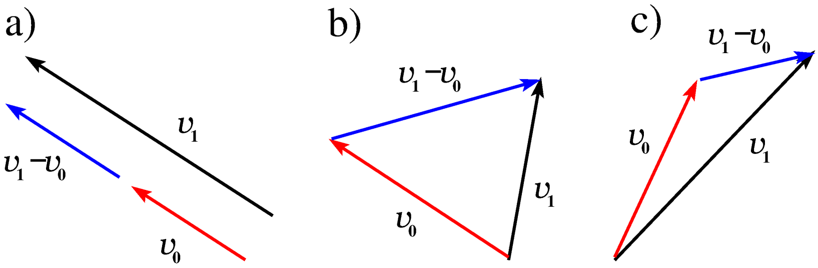


3-d axes in medicine



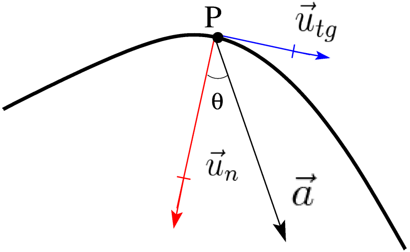


2-d acceleration



The three causes of acceleration: (a) Variation of amplitude (modulus) of velocity, but with the same direction (b) change of direction, but with the same modulus, (c) change in both direction and modulus. In all three cases v_0 represents the initial velocity, while v_1 is the final velocity.

Normal $\vec{a}_n$ and tangent $\vec{a}_t$ acceleration

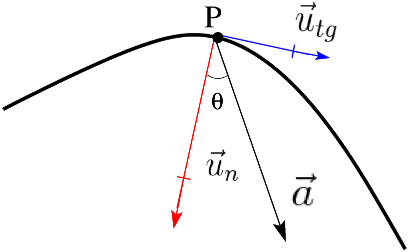


- tangential acceleration $a_t = |\vec{a}| \cos(\vartheta)$
- normal acceleration $a_n = |\vec{a}| \sin(\vartheta)$

$$a_t = \frac{d^2 s}{dt^2} \qquad a_n = \frac{1}{R}\left(\frac{ds}{dt}\right)^2$$

.

Normal $\vec{a}_n$ and tangent $\vec{a}_t$ acceleration



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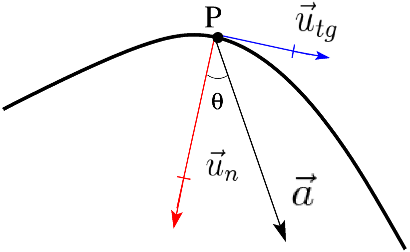

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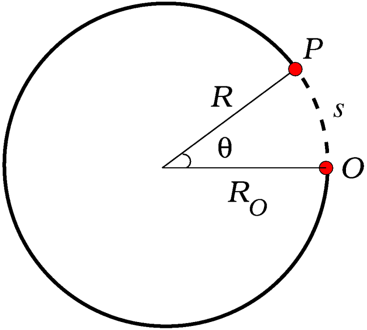
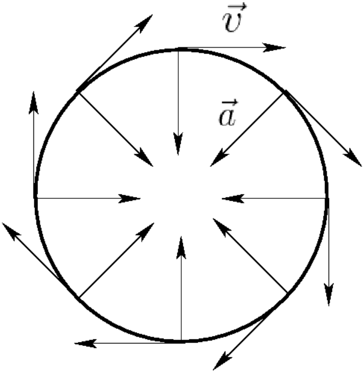
Uniform circular motion

Motion of a point on a circumference with constant speed modulus.





Acceleration is constant in module and always directed towards the center.
Velocity vectors have tangential directions and the same module.

Uniform circular motion

If θ is the angle

$$s\,(t) = R \cdot \theta(t)$$

being R constant

$$v = \frac{d\,s}{d\,t} = R\frac{d\,\theta}{d\,t}$$

$\frac{d\,\theta}{d\,t}$ is angular speed (pulsation) ω :

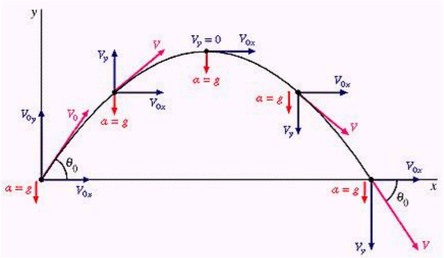
$$\frac{d\,s}{d\,t} = \omega\,R$$

and

$$|\vec{a}| = \frac{1}{R}\omega^2 R^2 = \omega^2 R$$

Balistic motion (Galileo)

- decomposition of the motion in two independent motions: one horizontal and one vertical.
- Being $\vec{a} = -g \cdot \hat{u_y}$, its components are: $a_x = 0$ and $a_y = -g$.
- The velocities: $v_x(t) = v_{x0}$ $v_y(t) = v_{y0} - g \cdot t$
- the x and y components of the motion correspond to a uniform and an accelerated rectilinear motions



Balistic motion

$$x(t) = x_0 + v_{x0} \cdot t \qquad y(t) = y_0 + v_{y0} \cdot t - \frac{1}{2} g \cdot t^2 \tag{1}$$

where x_0 and y_0 are the coordinates of the initial position (at $t = 0$). To derive the trajectory we isolate t (from the first eq):

$$t = \frac{x - x_0}{v_{x0}} \tag{2}$$

and substitute in (1):

$$y(x) = y_0 + v_{y0} \cdot \frac{x - x_0}{v_{x0}} - \frac{1}{2} \cdot g \cdot \frac{(x - x_0)^2}{v_{x0}^2} \tag{3}$$

Balistic motion

At the point of maximum height (the apex of the parabola) $v_y = 0$, and the corresponding time is given by $t_a = \frac{v_{y0}}{g}$; replacing this value in Eq:1 we obtain:

$$x_a = x_0 + v_{x0} \cdot \frac{v_{y0}}{g} \quad y_a = y_0 + \frac{v_{y0}^2}{2g}. \tag{4}$$

Speed modulus: v_x e v_y: $v = \sqrt{v_x^2 + v_y^2}$

The speed direction is tangent to the trajectory.

Balistic motion

If the initial velocity is given by its module v_0 and the elevation angle value θ_0, the Cartesian components are located via the axis projections:

$$v_{x0} = v_0 cos(\theta_0) \quad v_{y0} = v_0 sin(\theta_0)$$

$$tan(\alpha) = \frac{v_y}{v_x}$$