Work and Energy

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DIMES

Work and Energy

Energy

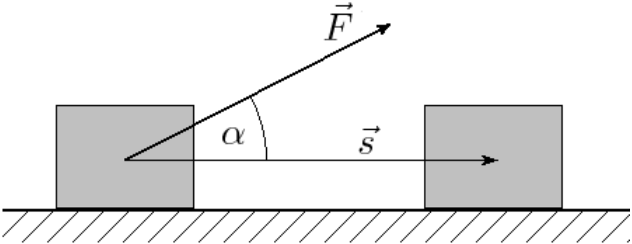
- Energy, work and power are often associated with engineering applications related to the functioning and performance of machines built by man.
- However, both from a historical point of view, both in application and in theory, these concepts have always been widely used in Medicine and Biology.
- The principle of conservation of energy was introduced, in its first formulation, by Julius von Mayer, physician and physicist of the nineteenth century, who is to be counted among the founders of Thermodynamics.
- The concept of potency is used in Physiology to quantify cardiac and metabolic activity, and many diseases can lead to imbalances in the production and consumption of energy by organisms (eg the relationship between arterial hypertension and heart disease has a purely mechanical origin).

Energy and work

- energy is perhaps the most important concept in physics and science
- its existence was definitely tested in 1850 (Joule)
- Energy in physics (and in biology) has a fundamental role: we can see every interaction (gravitational, electromagnetic, thermodynamic transformation) as energy exchanges
- Mechanics and thermodynamics can be formulated in terms of energy

Work: $L = \vec{F} \cdot \vec{s} = |\vec{F}| \cdot |\vec{s}| \cdot cos(\alpha)$

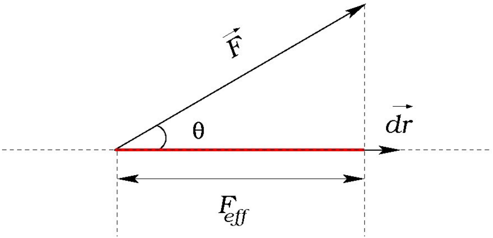
Work is associated with the effort required to move an object. The larger the object or the move, the greater the effort. If an object is stationary and a force acts on it, the work performed on the object is zero, even if we are making an effort.



Work

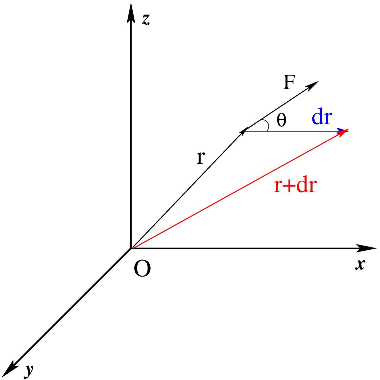
The sign of the work L depends on $\cos(\vartheta)$:
If $0 \leq \vartheta < \pi/2 \Rightarrow \cos(\vartheta) > 0 \Rightarrow \vec{F} \cdot \vec{s} > 0$ active work.
If $\pi/2 < \vartheta \leq \pi \Rightarrow \cos(\vartheta) < 0 \Rightarrow \vec{F} \cdot \vec{s} < 0$ resisting work.
If $\vartheta = \pi/2 \Rightarrow \cos(\vartheta) = 0 \Rightarrow \vec{F} \cdot \vec{s} = 0$ null work.

- Work is the product of the displacement module for the projection of the force in its direction (effective force).



Infinitesimal work: $dL = \vec{F}(\vec{r}) \cdot d\vec{r}$

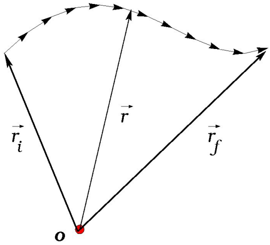
A material point that, at a certain moment, is in the $\vec{r}$ position within a force field, is subject to a force $\vec{F}\,(\vec{r})$.



- Infinitesimal work = scalar quantity:
- $$dL = \vec{F}(\vec{r}) \cdot d\vec{r}$$
  $$= |\vec{F}(\vec{r})| \cdot |d\vec{r}| \cdot cos(\theta)$$

Work along a trajectory = integral

Material point that moves from an initial position $\vec{r}_i$ to a final position $\vec{r}_f$ along a trajectory TR.



$$L\left(\vec{r}_i \stackrel{TR}{\rightarrow} \vec{r}_f\right) = \int_{\vec{r}_i}^{\vec{r}_f} dL = \int_{\vec{r}_i}^{\vec{r}_f} \vec{F}(\vec{r})d\vec{r}$$

Work

- Work is a scalar quantity
- Its dimensionality is $[F \cdot l] = [ML^2T^{-2}]$
- its IS measurement unit is Joule J
- Other measurement units:
- 1 ElettronVolt= $e \cdot V$=1.6 $\times 10^{-19} J$
- Calorie : 1 Cal=4.184 J
- erg : $1erg = 1dina \times 1cm = 10^{-5}N \times 10^{-2}m = 10^{-7}J$

Kinetic energy

Definition

Kinetic energy of a point mass:

$$E_C = \frac{1}{2} m v^2$$

m is the mass and v its velocity

$$L\left(\vec{r}_i \overset{TR}{\rightarrow} \vec{r}_f\right) = E_C\left(\vec{r}_f\right) - E_C\left(\vec{r}_i\right) = \frac{1}{2} m v_f^2 - \frac{1}{2} m v_i^2$$

where v_f and v_i are the mass speed in $\vec{r}_f$ and $\vec{r}_i$. Thus a non null works changes kinetic energy

Fundamental kinetic energy theorem

- **The change in kinetic energy is due to work performed by a force**
- an object moving along a trajectory might change its speed due to the action of forces
- the change in kinetic energy is equal to the total work applied on the object the forces on the trajectory of motion

Fundamental kinetic energy theorem

Equivalence between work and change of kinetic energy

Demonstration (not rigorous)

$L = \vec{F} \cdot \vec{s}$, fron the second Principle of dynamics $\vec{F} = m\vec{a}$, thus $L = m\vec{a} \cdot \vec{s}$.
For uniformly accelerated motion:

$$\begin{cases} v = at \\ s = \frac{at^2}{2} \end{cases}$$

we get $s = \frac{v^2}{2a}$, thus work on mass m can be written as:

$$L = \frac{mv^2}{2}$$

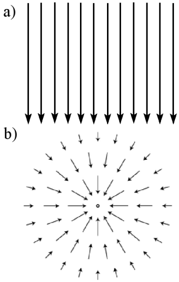
Kinetic energy E_C and work L

- Even in the case of an unknown force, we can calculate the work done by the kinetic energy variation.
- If the kinetic energy does NOT change, the force DOES NOT perform work.
- Example: centrifugal force does not perform work
  - force is always perpendicular to displacement
  - in uniform circular motion the velocity module is constant

Potential energy

Force field

Force field is a zone of space in which it is possible to associate a force vector at each point. A material point that moves in a field is subject, at any point, to a force: $\vec{F} = \vec{F}(\vec{r})$



a) Constant field (eg weight force, condenser)

b) Gravitational field around the earth

A $\vec{F}$ **uniform** field does not depend on the position (and time): it is a constant vector in module, direction and verse at each point of the space:

$$\vec{F} = \overrightarrow{cost}$$

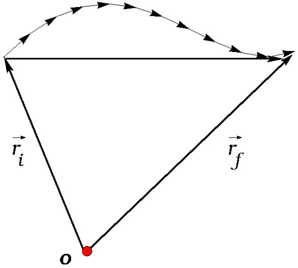
Example: $m\vec{g}$ is uniform.

Work of a conservative force

In general, changing the trajectory from $\vec{r}_i$ to $\vec{r}_f$ changes also work L. There exist force fields which work, fixed $\vec{r}_i$ and $\vec{r}_f$, do not depend on trajectory: **conservative** fields. A uniform force $\vec{F}_K$ is conservative:

$$L\left(\vec{r}_i \stackrel{TR}{\rightarrow} \vec{r}_f\right) = \int_{\vec{r}_i}^{\vec{r}_f} \vec{F}(\vec{r}) d\vec{r} = \vec{F}_K \cdot \int_{\vec{r}_i}^{\vec{r}_f} d\vec{r} = \vec{F}_K \cdot \Delta\vec{r}$$

$\int_{\vec{r}_i}^{\vec{r}_f} d\vec{r}$ is the total displacement $\vec{r}_f - \vec{r}_i$ (sum of infinitesimal displacements)



For gravitational forces

$$L\left(\vec{r}_i \stackrel{TR}{\rightarrow} \vec{r}_f\right) = m\vec{g} \cdot (\vec{r}_f - \vec{r}_i)$$

every reference to trajectory disappears

NOTE on fields and forces

- As we shall see, associated with each force there is also a **field** (eg electric field, $E = F/q$, gravitational field g, magnetic field B).
- To define conservativity (and therefore the existence of a potential function) it is necessary to look at whether the work performed by the field (called **circuitation**) depends on the route or not.
- In many cases, field and force are “the same thing” (same direction, then same product scalar with the shift Δs) so no distinction is made.
- In some cases, however, there is a difference (magnetic field).

Conservative field

- If a field is conservative work does not depend on the trajectory but only from $\vec{r}_f$ and $\vec{r}_i$.
- Work can always be written as the difference between the value of a function in $\vec{r}_i$ and in $\vec{r}_f$.
- This function is called "potential energy".

$$\begin{aligned} L\left(\vec{r}_i \overset{TR}{\to} \vec{r}_f\right) &= m\vec{g}\cdot\left(\vec{r}_f - \vec{r}_i\right) \\ &= m\vec{g}\cdot\vec{r}_f - m\vec{g}\cdot\vec{r}_i \\ &= \left(-m\vec{g}\cdot\vec{r}_i\right) - \left(-m\vec{g}\cdot\vec{r}_f\right) \end{aligned}$$

$$L\left(\vec{r}_i \to \vec{r}_f\right) = mg\cdot\left(z_i - z_f\right).$$

The function $U\left(z\right) = mgz$ is called gravitational potential energy

Potential energy

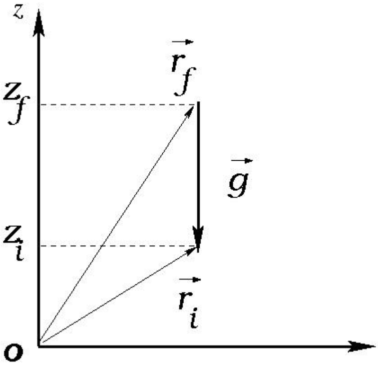
- Potential energy $U(\vec{r})$ is a scalar function of position
- Represents the **ability to perform work** that the body possesses in virtue of its position within a conservative force field.
- is a status function (minima give the positions of **equilibrium)**
- **The potential energy is defined up to an additive constant.**

In a conservative force field if a body moves from r_A to r_B, the field forces perform a defined work on it.

$$L = -\Delta U = -[U(\mathbf{r}_B) - U(\mathbf{r}_A)]$$

L depends only on r_A and r_B.

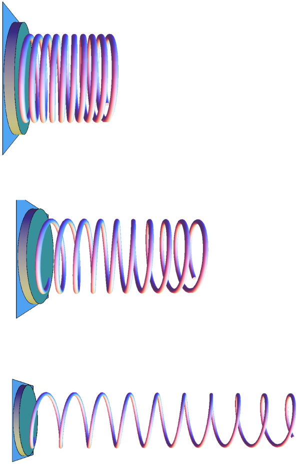
Gravitational potential energy



For the calculation of gravitational potential energy, we can consider projection along the z axis of the position vectors

$$h = z_f - z_i$$

Springs and elastic force



$$F = -k(x - l_0)$$

$$\text{se} \quad l_0 = 0 \Rightarrow F = -kx$$

A spring has different values of potential energy

$$U = \frac{k}{2}x^2$$

$$(\Delta U = \int F \cdot ds = \int kx \cdot dx)$$

$$\int x dx = \frac{x^2}{2}$$

Other fields

- Weight force $m\vec{g}$ has a potential energy $U(h) = mgh$.
- Gravitational force $G\frac{m_1 \cdot m_2}{r^2}$ has a potential energy $U(r) = -G\frac{m_1 \cdot m_2}{r}$.
- Coulomb electrostatic force $\frac{1}{4\pi\epsilon_0}\frac{q_1 \cdot q_2}{r^2}$ has a potential energy $U(r) = \frac{1}{4\pi\epsilon_0}\frac{Qq}{r}$.
- elastic force $F = -kx$ has a potential energy $U(x) = \frac{1}{2}kx^2$

Mechanical energy

Conservation of Mechanical energy

- if a material point moves in a conservative field, the sum of the kinetic energy and the potential energy of the point remains constant over time
- The sum of kinetic energy and potential energy is called "total mechanical energy"
- In a conservative field the total mechanical energy of a material point is conserved (Mechanical energy conservation theorem).

Example in gravitational field

$$E = \frac{1}{2}mv^2 + mgz = costant$$

The constant has no physical meaning

Conservation: $E_M = E_C + U$

$$L = -\Delta U = -[U(\mathbf{r}_B) - U(\mathbf{r}_A)]$$

or, from the fundamental theorem of kinetic energy:

$$L = \Delta E_C = [E_C(\mathbf{r}_B) - E_C(\mathbf{r}_A)]$$

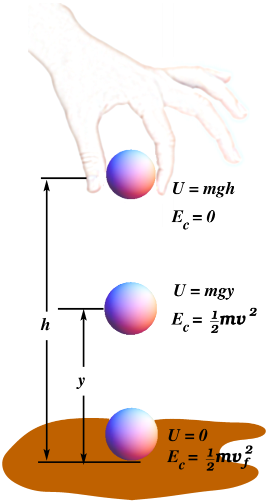
thus

$$-\Delta U = \Delta E_C \Rightarrow E_C(\mathbf{r}_A) + U(\mathbf{r}_A) = E_C(\mathbf{r}_B) + U(\mathbf{r}_B)$$

or

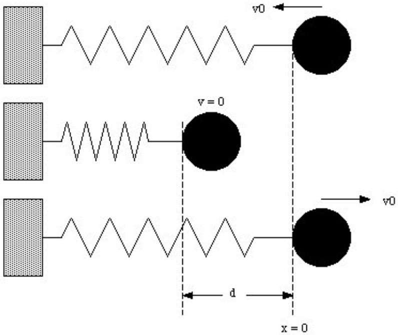
$$E_M(\mathbf{r}_A) = E_M(\mathbf{r}_B)$$

Free falling weight: $v = \sqrt{2gh}$



- At the beginning the energy is all potential mgh.
- During the fall it becomes more and more kinetic $\frac{mv^2}{2}$.
- The instant it touches the ground is all kinetic.
- $mgh + 0 = mgy + \frac{1}{2}mv_f^2$
- $v_f = \sqrt{2g\,(h-y)}$

Compressed spring: $v = \sqrt{\frac{kx^2}{m}}$



- At maximum compression and elongation the energy is all potential $\frac{kl^2}{2}$.
- During the passage along the rest position the energy is all kinetic $\frac{mv^2}{2}$.
- In the points of maximum elongation and compression there is inversion of the motion $E_C = 0$.
- The zero of the potential is taken in the rest position.

relation between force and potential energy

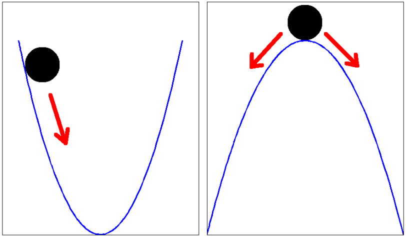
$$\vec{F} = -\vec{\nabla}U = (-\frac{dU}{dx}, -\frac{dU}{dy}, -\frac{dU}{dz})$$

$$U(r) = -\int F(r)dr$$

- The first formula means that the force **pushes** towards the lows of potential energy.
- The force is canceled when the slope of the graph is zero, i.e. when it becomes horizontal (maximum or minimum of the potential function U).
- Given the total energy of the system, this will position itself in the position of minimum potential value.

Potential energy minima

The minimum points of potential energy correspond to positions of stable equilibrium, while those of maximum to positions of unstable equilibrium



We can see the spring's equilibrium position as the state of minimum potential energy: each perturbation (applied force) will move the body away from the minimum position, to which it (for the relation between potential energy and force) will tend to come back spontaneously.

U minima and elastic potential

- The region close to the minimum of any potential can be approximated by an elastic potential.
- This is why the “small perturbations” of a state of equilibrium produce an approximately harmonical motion, and this condition applies in general.

Power

Power: $P = \frac{\Delta L}{\Delta t}$

- Time does not appear in the work definition: there is only force and displacement.
- The work depends only on the way in which the point passes from the initial position to the final position and not from the time taken.
- To make explicit the temporal dependence, we introduce the power: $P = \frac{\Delta L}{\Delta t}$
- Measurement unit is Watt W: $\frac{\text{joule}}{s}$
- The power measures the amount of energy exchanged in the unit of time, in any transformation process: mechanical, electrical, thermal or chemical.

Other Common Measurement units

kW,kWh, CV, HP (1 CV = 735 W ; 1 CV = 0,9863 HP)
unit conversions:

joule(J)	calorie(Cal)	KWh(kW0ra)
1	0.2389	2.778×10^{-7}
4.186	1	1.163×10^{-6}
3.6×10^{6}	8.6×10^{5}	1

Power and velocity

If a constant force $\vec{F}$ acts on a body causing a shift $\Delta\vec{r}$ in a time Δt, the average power is given by:

$$P = \frac{\Delta L}{\Delta t} = \vec{F} \cdot \frac{\Delta\vec{r}}{\Delta t} = \vec{F} \cdot \vec{v}_M$$

Instantaneous power:

$$P = \vec{F} \cdot \frac{d\vec{r}}{dt} = \vec{F} \cdot \vec{v}.$$

Basal metabolic rate: $P = \frac{\Delta L}{\Delta t}$

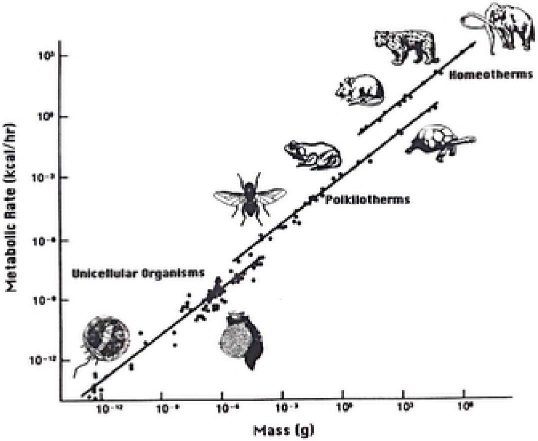
Basal metabolic rate is the amount of energy spent per unit of time, in a constant temperature environment, in the post-digestive state (after 12 hours of fasting). This energy is sufficient for the functioning of the resting vital organs.

$$P = \left(\frac{10.0m}{1\ \text{kg}} + \frac{6.25h}{1\ \text{cm}} - \frac{5.0a}{1\ \text{anno}} + s \right) \frac{\text{kcal}}{\text{day}}$$

m is the mass, h is height a is age. s is +5 for men and -161 for women

For example, a 55-year-old woman weighing 59 kg and 168 cm tall has a basal metabolic rate of 1272 kcal per day or 53 kcal / hour (61.3 watts).

Metabolic rate in animals: scaling laws

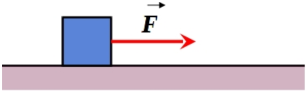


Metabolic rate approximately proportional to mass (volume)

Exercise

- A 60 kg block is placed on a perfectly smooth surface. If we apply a $\vec{F}$ constant force of 100 N and parallel to the table, for a time t of 3 seconds:
  - Which final speed?
  - Which work done?
- A $\vec{P}$ weight body slides along an inclined plane with no friction along l. What is the work done by the weight force?
- What work does a 70kg man do to climb the top of the Asinelli tower (h = 100m)?

Solutions



$$v = v_0 + a \cdot t = \frac{F}{m} \cdot t$$

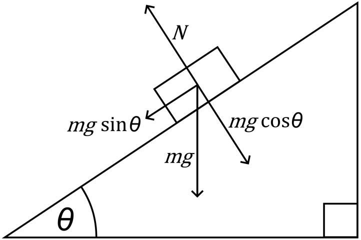
$$= \frac{100N}{60kg} \cdot 3s = 5\frac{m}{s}$$

$$v^2 = 2as \Rightarrow s = \frac{v^2}{2a}$$

$$L = F \cdot s = 100N \cdot 7.5m = 750N$$

$$L = \Delta E_c = m\frac{v^2}{2} - 0 = 750N$$

Solutions

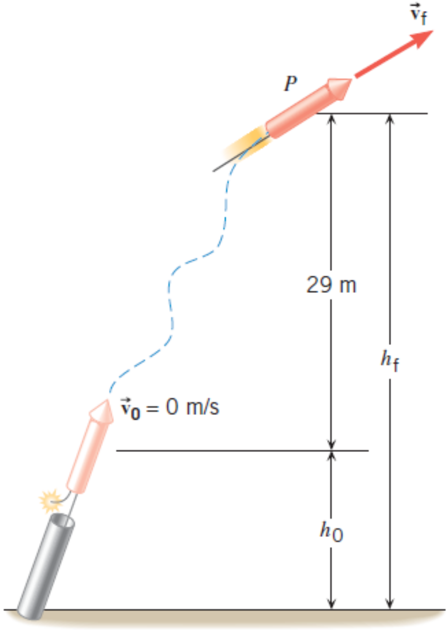


$$L_D = mg \cdot l \cdot sin(\theta)$$

$$L_B = mg \cdot b \cdot cos(\pi/2) = 0$$

$$L_H = mg \cdot h \cdot cos(0) = mgh$$

A 0.20 kg rocket in a fireworks display is launched from rest and follows an erratic flight path to reach the point P, as shown in the figure.
Point P is 29 m above the starting point. In the process, 425 J of work is done on the rocket by the nonconservative force generated by the burning propellant.
Ignoring air resistance and the mass lost due to the burning propellant, find the speed vf of the rocket at the point P.


$\vec{v}_f$
P
29 m
h_f
$\vec{v}_0 = 0$ m/s
h_0

applying the kinetic energy theorem, we know that the variation in kinetic energy is the sum of all the works applied by all the forces.
Given that the rockes start from still, the kinetic energy variation is equal to the final value $\Delta E_k = E_{kf} = \frac{1}{2}mv_f^2$
The total work performed on the rocket is the positive work from the fuel burning and the negative one from the gravity (since it increased it's vertical position).
Therefore: $\Delta E_k = 425 - m \cdot g \cdot h$ and replacing all the values we obtain: $\frac{1}{2}mv_f^2 = 425 - m \cdot g \cdot h$
$v_f = \sqrt{\frac{2}{m}(425 - m \cdot g \cdot h)}$