Rigid Body Dynamics

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DIMES

Systems of points

Systems

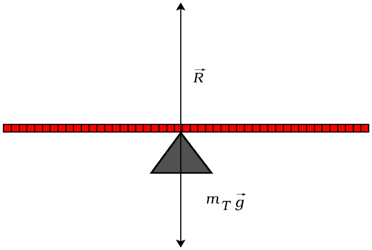
- the third law of dynamic and the conservation of momentum and energy are the foundations for studying systems composed of multiple point particles
- The various states of aggregation of matter (solid, liquid, gas) can be described as systems composed of point particles with different characteristic interactions
  - liquids: interaction that can be approximated as short range attraction
  - perfect gas: elastic collisions between the atoms and the container walls (and no attraction between the atoms)
  - solid: fixed (or almost fixed) and elastic interactions between neighbours atoms (that generate oscillation and rotations around these fixed positions)

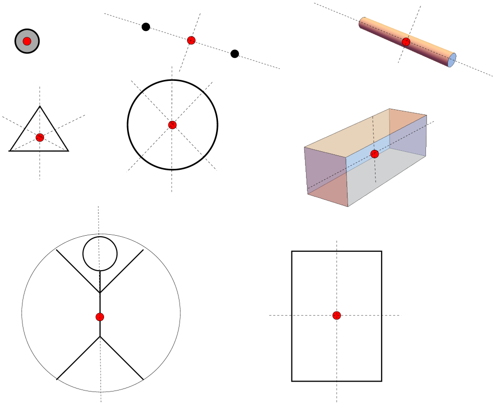
- the concepts of systems of points will be the topic of this lesson, the fluid ones and the ones abut thermodynamics
- in the thermodynamics ones we will describe how these systems can exchange energy with the environment as either work of heat
- this lesson will focus on the rigid solid bodies: these are ideal object where the internal distances between atoms do no change
- Rigid solids have a well defined shape, volume and cannot be deformed, contrary to the liquids (that have a proper volume but not a proper shape) and gases (that have neither a fixed volume or shape)
- the usefulness of the rigid body approximation is that it is very easy to describe: we don't need to specify all the positions of all the atoms, but only the position of the center of mass and how the body is rotated in space (3 more numbers)

- Some properties of the rigid body are extensions of the single point particle
  - the mass of N point particles $m_i, i = 1, \dots N$ is the sum of the individual masses: $M = \sum_{i=1}^{N} m_i$
  - linear momentum of the rigid body is given by the sum of all the individual linear momentums of each individual point particle $\vec{Q} = \sum_{i=1}^{N} q_i$
  - the energy is given by the sum of the potential and kinetic energies of each point particle.
- We can also define the average properties of the system, by averaging the properties of each individual point particle
- This is the way we defined the center of mass, that we know from previous lessons, will move in a uniform rectilinear motion in a isolated system

- In the rigid body motion it can not be ignored the finite size and the mass distribution of the particles inside it
- we can define the center of mass as the weighted average of all the masses inside the body
- any force applied on the body can be divided in two terms:
  - one that moves the center of mass but does not rotate the body
  - one that rotates the body around an axis going through the center of mass but does not move the center of mass
- we can define the center of gravity as the center of all the forces applied on the body, so that there is no torque making it rotate.
- in the approximation of the uniform field, these center of mass and the center of gravity coincide, but this is not true in general. for example, in skyscrapers like the Empire State Building, the two are different by 1 mm, due to the difference in gravity between the top and the bottom floors.

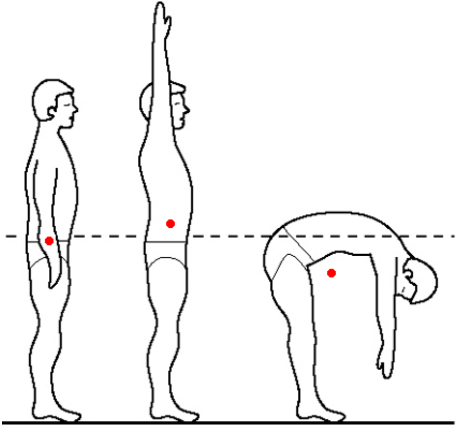
- If a rigid bar is placed on top of a hinge point (the fulcrum), there will be a reaction force of the fulcrum on the bar.
- if the fulcrum is under the center of mass, there is no motion.
- If it is not, there will be a rotation around the fulcrum (and a translation of the center of mass)
- for solid objects composed of continuous matter instead of point particles, the center of mass is determined by the distribution of the masses: it will be an integral instead of a summation





- The exact position of the center of mass can be often found using geometrical consideration for simple bodies, such as geometrical shapes.
- Given that most objects can be approximated as a union of simple shapes, this can be used also for more complicated objects.

Variations of the center of mass (CoM) with body posture

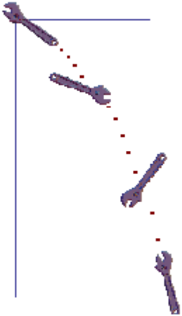


- standing: CoM inside the body
- bending over: the CoM is outside the body
- an example: in high jumping (Fosbury) the CoM does not need to go over the bar and bending over allow to surpass higher bar. This is not the case of a hurdles race

Translational and rotational motions

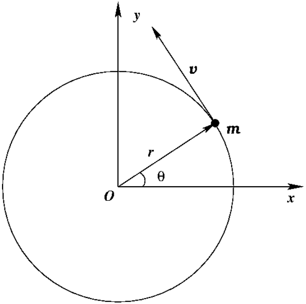
- Rotations can be described with a similar framework of linear motion of the point particle
- Some of these equivalent quantities are:
  - the torque, equivalent of the force
  - angular speed, equivalent to the linear ones
  - the angular momentum, equivalent of the linear momentum
  - rotational inertia, equivalent of the mass
- it is possible to write laws similar to the principles of dynamics of the point particle

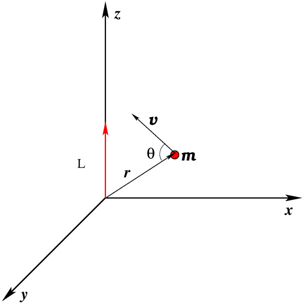
Motion of the rigid body



- we can divide the motion of the center of mass and the rotation around it
- the rotation speed is an angular speed (ω similar to the one defined in uniform rotations)
- the goal is to be able to describe rotation quantities equivalently to the linear ones

Angular momentum: $\vec{L} = \vec{r} \times \vec{p} = \vec{r} \times m\vec{v}$





- the direction of the angular momentum is given by the cross product expression
- counterclockwise rotation on a xy plane generate a vector pointing toward the up direction of the z axis (right hand rule).
- clockwise rotation will be directed toward the negative values of z

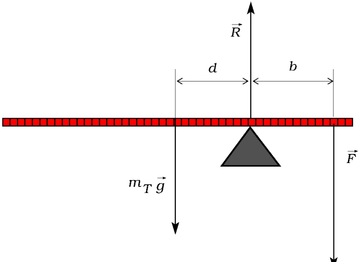
Biomechanics

Axis of rotation

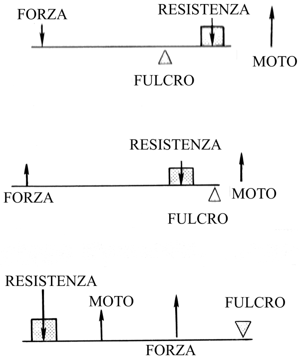
- at any moment in time, given the motion of the body, we can identify a single axis of rotation, called Mozzi's axis
- in the simplest motions this stays constant in time
- in general it will change over time
- this is valid also for motion without any force applied to it: choosing a point, we can calculate it's angular momentum relative to that point

Rotational equilibrium of a rigid bar

- if the fulcrum is not under the CoM, we need to apply a force F to stop the rotation
- each force applies a torque equal to the force, multipled by the distance from the fulcum, with a sign that depends on the rotation direction
- there is equilibrium when the torque applied by both the force F and the weight are equal and opposite
- $Fb - m_T g d = 0 \quad \Rightarrow \quad F = m_T g \frac{d}{b}$

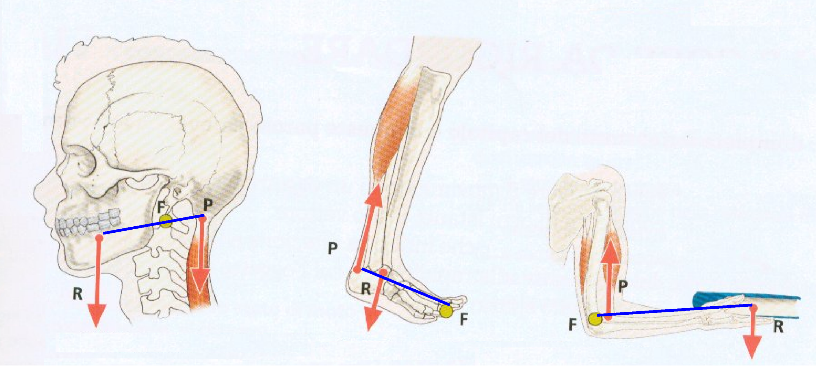


Levers



- two forces are applied on the rigid body, tipically the rigid body's weight
- **first class**:the fulcrum is between the two forces. the can be neutral, advantageous or disadvantageous
- **second class**: the weight is between the fulcrum and the applied force, they are always advantageous
- **third class**: the applied force is between the fulcrum and the weight, they are always disadvantageous

Levers in the human body: joints



I class II class III class

Levers in the human body: joints optimal range

- the torque that is applied depends on the direction of application
- depending on the position of the joint, the effectiveness of the muscle changes.
- for example, in the biceps, the position of the arm around the 90° angle is the optimal one
- the extended arm is the least effective one
- this is why range of motion is as important as weight load, for example when exercising for muscle recovery

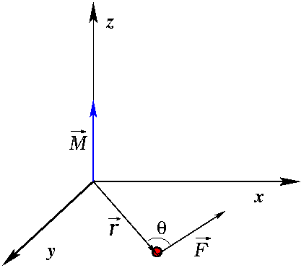
square-cube law

- the strength of a muscle (or the resistance of a bone) are proportional to their section.
- the weight of the bone or muscle is proportional to their volume.
- if everything get doubled in size, the resistance increase 4-fold, but the weight increases 8-fold.
- This means that the bigger the body, the harder is it to move.
- This is why elephants can't jump!

Torque and rotations

Torque

- extend the concept of torque found in levers to any force applied on the body
- vectore:$\vec{M} = \vec{r} \times \vec{F}$ with magnitude:$|\vec{M}| = |\vec{r}| \cdot |\vec{F}| \cdot sin(\theta)$
- dimensions: $F \cdot l$=ML^2T^{-2}
- IS unit of measurement: $N \cdot m/rad$

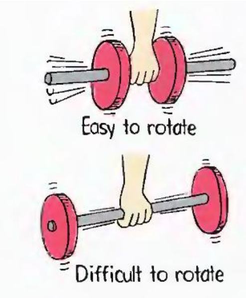


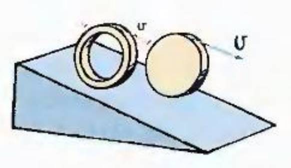
Rigid body equilibrium

- the equilibrium is the absence of any motion: it is defined both in terms of translation and rotation
- there is equilibrium if the summation of all the forces and all the torques are null

$$\begin{cases} \sum_{i=1}^{N} \vec{F}_i = 0 \\ \sum_{i=1}^{N} \vec{M}_i = 0 \end{cases}$$

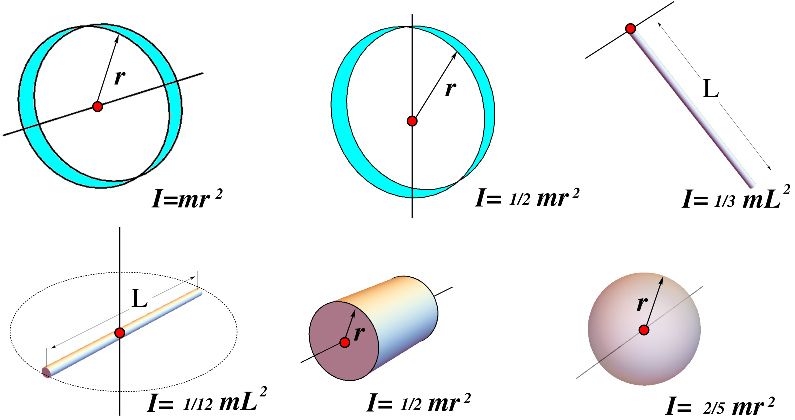
Rotational Inertia $\Leftrightarrow$ Linear Inertia (mass)





- is the resistance the body applies to changes to its state of rotations
- for rotations there is the equivalent of the first principle
- the rotational inertia depends on the distribution of matter
- this quantity is referred to as rotational inertia I
- in general it is complicated to calculate, as it depends around which axis are we trying to rotate the body
- we can calculate around a specific axis: $I_{A.R.} = \sum_{i=1}^{n} m_i r_i^2$

Moments of Inertia



rotation around a fixed axis

- if all the rotations are happening around a specific axis, we can write the equivalent of the second principle of dynamic
- if we call the angular velocity ω and the angle of rotation θ, then

$$I\frac{d\omega}{dt} = M$$

- if the torque is constant we have an equivalent time law to the linear accerelated motion

Rotational work

is defined in a similar way to the work done by translation:

$$W = \int_{\theta_1}^{\theta_2} \vec{M} d\vec{\theta}$$

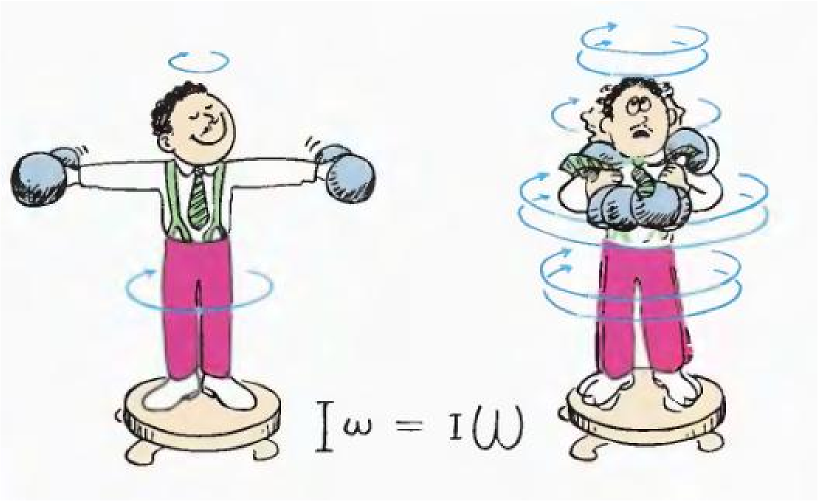
if the rotation and the torque are on the same axis, it becomes the product of the magnitudes, with sign depending on the sense of rotation

$$W = \int_{\theta_1}^{\theta_2} |M| d\theta$$

we can define a rotational energy in a similar fashion to the kinematic energy:

$$E_r = \frac{1}{2} I \omega^2$$

Angular momentum conservation: $\vec{L} = I\omega$



Angular momentum conservation

- Angular momentum is a quantity that is conserved like linear momentum
- this property comes from the third principle of dynamics
- linear and angular momentum conservation, as well of energy conservations, are special cases of the Noether teorem
- this theorem links simmetries in our systems with conserved quantities
- energy $\Leftrightarrow$ time progression
- linear momentum $\Leftrightarrow$ space omogeneity
- angular momentum $\Leftrightarrow$ space isotropy

exercises

holding a weight

- A person holds a 500 Newton (N) dumbbell in their right hand.
- Their forearm and hand are held stationery in the horizontal position with no rotation at the elbow joint.
- The forearm and hand segment weighs 17 N, and the center of gravity of the forearm/hand segment is 0.23 meters (m) from the axis of the elbow joint.
- The center of gravity of the dumbbell is 0.34 m from the elbow joint.
- If the muscle holding the arm in this position inserts 0.05 m from the elbow joint, how much muscle force is required to keep the forearm/hand from rotating at the elbow joint?

holding a weight - solution

- to keep the arm from moving, the sum of all the torques at the elbow must be 0
- the dumbell has a weight of 500 N, with a moment arm of 0.34 cm, so a torque of $-170.0\,Nm$
- the arm has a weight of 17 N, with a moment arm of 0.23 cm, so a torque of $-3.9\,Nm$
- the bicep must exert enough force to create a torque of $-((-170) + (-3.9)) = 173.9\,Nm$
- having a moment arm of 0.05 m, the force must be $F = M/r = 3'478N$

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rotating the shoulder

- let's consider the force applied by the shoulder muscles to rotate the arm swinging it around
- the arm weights around 6 kg
- it is long around 60 cm
- let's assume the rotation have a period of 1 second
- assuming that the muscles are attached at an average distance of 5 cm from the shoulder, how much force do they have to apply to make it swing that fast in 2 second from still?
- assume the muscle is attached perpendicular to the humerus (not true, just making this easy)

rotating the shoulder - solution

- the angular inertia is the one of a bar rotating around an extreme: $\frac{1}{3}mL^2 = \frac{1}{3}6 \cdot 0.6^2 = 0.72\, kgm^2$
- the angular speed is a rotation of 2π in 1 second, so it's $2\pi\, radians/s$
- the average torque to change from a speed of 0 to $2\pi\, radians/s$ in 2 seconds is $M = I\frac{\Delta\omega}{\Delta t} = \frac{1}{3}6 \cdot 0.6^2\frac{2\pi}{2} = 0.72\pi\ N \cdot m/rad$
- this is the torque that the muscle should apply: $M = F \cdot r \cdot \cos\theta \Rightarrow F = \frac{M}{r}$
- $F = \frac{0.72\pi}{0.05} = 45.2N$

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rotating the shoulder - extension

- how the problem would change if I was trying to trow an ball of mass 0.5 kg holding it in my hand?
- hint: consider the object as a point mass, and remember that the rotational inertia is additive
- the ball will add an additional $I_b = 0.5 \cdot 0.6^2 = 0.18\, kgm^2$, a 25% increase
- being the force required proportional to the torque, that itself is proportional to the rotational inertia, we can expect a 25% increase in the required force, so an increase of $11.3N$

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