Fluid Mechanics
Surface Tension, Diffusion, and Osmosis

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DIMES

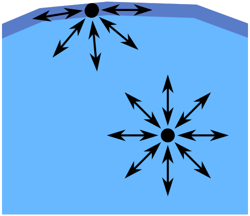
Surface Tension

Liquid surfaces

- Up to now we considered the liquids as completely enclosed in a pipe, with no free surfaces
- today we discuss what happens when we have some of these free surfaces
- the two main effects will be surface tension and gas diffusion
- they are both fundamental to understand the pulmonary and kidney behavior
- liquid surfaces can behave very differently from the bulk material

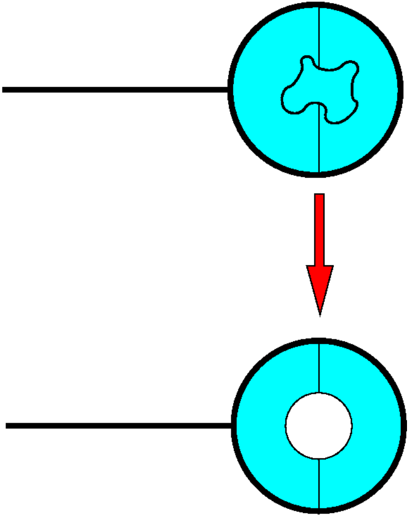
molecular attraction

- molecules in a fluid exert a light, short range attractive force with each other
- they also interact as rigid bodies, colliding like hard spheres due to their thermal kinetic energy
- in the bulk of the fluid these forces on average even out, so the net force is zero
- on the surface, on the other end, there are still all the attractive forces from the inside of the bulk, but not from the outside
- this lead to the formation of a net force toward the center of the bulk, called surface tension



surface tension

- due to these internal forces, the liquid assumes a preferred surface shape
- in normal condition it is just horizontal and flat
- in general, not under the effect of gravity, the shape that minimize the potential energy of the system is spherical



surface tension - work interpretation

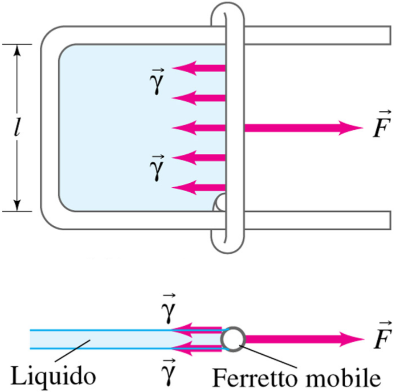
- surface tension is a characteristic of the interaction between the fluid and the material with which is in contact, for example water and air, of water and oil
- can be described as the work W necessary to change the amount of surface ΔA of the liquid once it is in equilibrium

$$W = \Delta A \cdot \gamma$$

- γ is referred as the surface tension

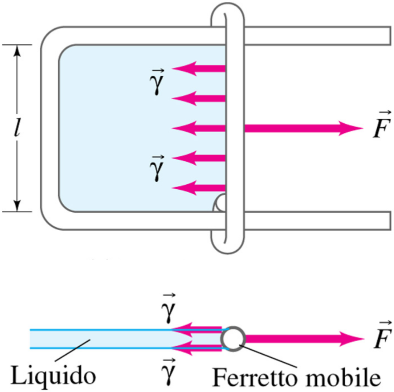
surface tension - force interpretation

- to generate a work is it necessary to apply a force,
- it can be viewed as the force that is necessary to constantly maintain the deformation
- it can also be viewed as the force that the liquid will apply to try and return to its previous state
- how to measure it?

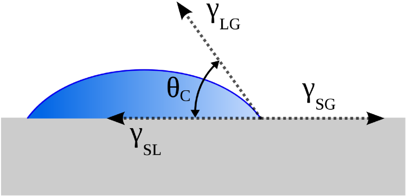


surface tension - force interpretation

- a movable bar on a rig filled with the liquid under study
- the liquid exert a force on the bar proportional to the length of the bar
- given that there are two surfaces applying force, each one contributes to half of the force $F = 2\gamma l$
- the work expression can be obtained by noticing that $W = F dx$ and $l \cdot dx = \Delta A$ where dx is a small movement of the bar



Young's law for drops



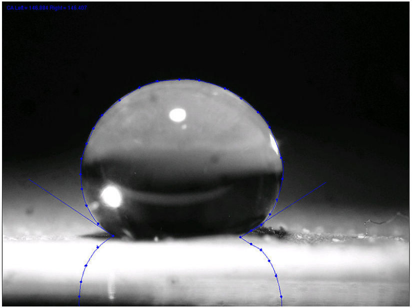
- if we want to describe a drop resting on a surface (let's say water on glass), we have three interfaces to consider:
  - γ_{SL} liquid-surface
  - γ_{SG} gas-surface
  - γ_{LG} liquid-gas
- we will observe that these 3 interfaces generates a net force that deforms the drop until a specific contact angle θ_C is reached, at which the three forces are in equilibrium

$$\gamma_{SG} - \gamma_{SL} - \gamma_{LG} \cos\theta_C = 0$$

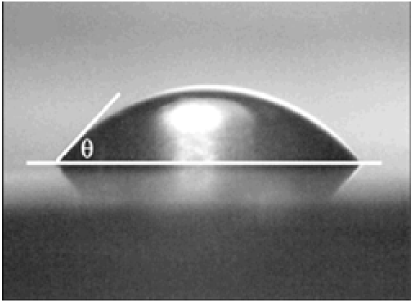
- if θ_C if less than 90° we say that the liquid is able to wet the surface, and there is a net force spreading the drop out

hydrophilic and hydrophobic substances

depending on the contact angle between a material and water in air, we talk about hydrophilic and hydrophobic materials



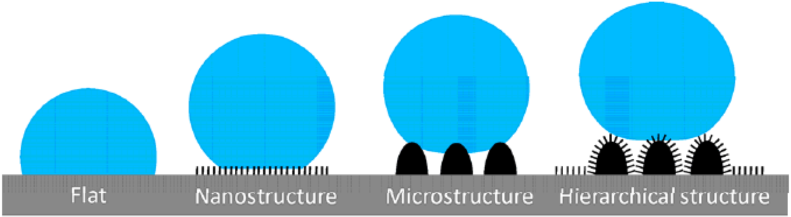
water on lotus leaf



water on glass surface

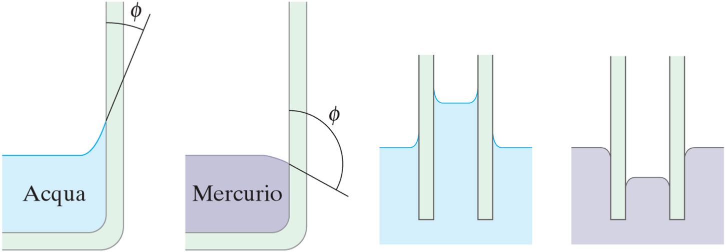
hydrophobic microstructuration

- using nanostructuration is is possible to transform a material that is normally hydrophilic in an hydrophobic
- is the method used by some plants like the lotus



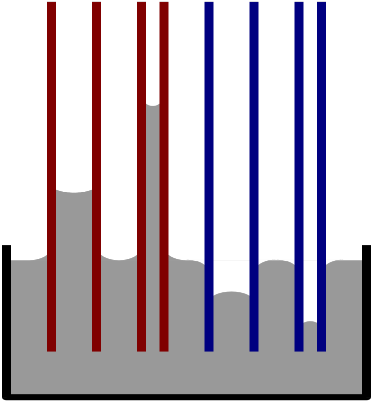
contact angle and capillarity

- the effect of capillarity is present whenever we put a liquid inside a vessel, but is noticeable only when the vessel size is small
- the contact surface between the vessel and the liquid apply a net force depending on the contact angle, in the same way as it does on a single drop
- the direction of the force depends on the contact angle



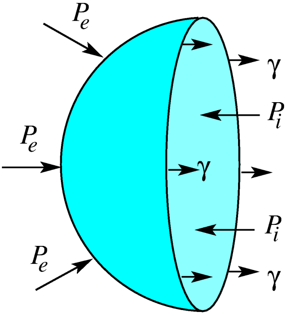
contact angle and capillarity

- this effect if bigger the greater the ratio of the contact surface to the volume inside the tube, so it's more prominent in capillars



Laplace's law

Laplace's law for bubbles



- when we have a bubble, the internal pressure of the bubble increases due to the tension of the free surface, that try to minimize itself
- there are 3 forces balancing out
  - the surface tension $\gamma 2\pi r$
  - the force due to external hydrostatic pressure $F_e = P_e \pi r^2$
  - the force due to internal hydrostatic pressure $F_i = P_i \pi r^2$
- at equilibrium $P_e \pi r^2 + \gamma 2\pi r = F_i = P_i \pi r^2$

$$\Delta P = \frac{2\gamma}{r}$$

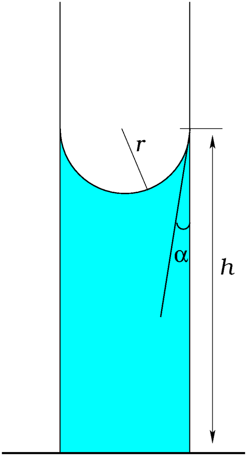
small bubbles and blood flow

- normally bubble can move freely in large vases
- when they move in smaller one they will increase their size due to the drop in pressure from the Venturi effect
- once they arrive in small enough vases, the pressure in the vase might not be enough to squeeze them in the size they need to be to go though the vase: this might lead to gas embolism, where a vase get closed by a bubble
- this typically happens only in very small vases, so a single small bubble is rarely an issue, but if enough of them are formed (for example due to breakage of a bigger one) entire sections of tissue might not get enough blood flow to keep it alive

Borelli-Jurin's law of capillar action

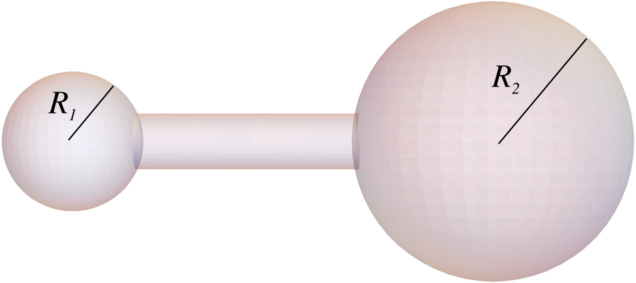
- if we include the net force effect of the surface tension and the gravity pull, we can find an equilibrium between the two
- to do so we have to use an approximation: the size of the capillary is smaller than the curvature that is due to the surface tension, so the surface can be approximated as a circular bubble
- in this case the tension can be approximated using the Laplace's law for bubble, giving the following result:

$$\rho g(h\pi r^2) = \gamma(2\pi r)$$



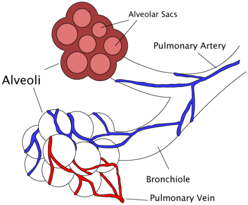
Laplace's law for two connected bubbles

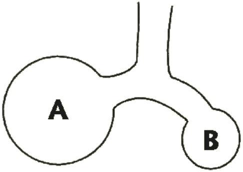
- according to Laplace's law the smallwer bubbles have a higher internal pressure.
- this leads to the un-intuitive effect that if we connect two bubble with different radius, the bigger one is going to get more air from the smalle one, due to the latter's higher internal pressure



Pulmonary alveolus

- this is a problem for pulmunary alveolus: they are all connected and stretched by the pressure inside the lungs
- without any external help they would "cannibalize" each other, leading to a loss in functionality
- to prevent this the body coats them in substances called surfactant, that reduce the surface tension of the alveoli
- surfactants are also one of the key components of detergents, easing the division of the oils in smaller droplets, that can then be brought in solution in the water





Diffusion

Diffusion

- Diffusion happens everytime there is a dishomogeneous concentration of a substance in a medium, and this is free to move around
- over time the concentration will even out all over the space (if there are no new sources or removal)
- this is referred as passive diffusion
- For each point in the medium we can study how the concentration changes over time, and how that behaves over surfaces
- each substance in that medium will diffuse more or less rapidly depending on the diffusion coefficient D

Fick's first law

- we can describe the concentration of the substance point by point with the letter ϕ
- Given an ideal surface in our medium, the amount of substance that will go throught in a unit of time per unit of surface is defined as the flux J

$$J = -D\frac{\Delta\phi}{\Delta x}$$

- the diffusion coefficient D is defined as an area over a time, so usually square meters over seconds

diffusion across a membrane

- If we do not consider a continuous space, but the diffusion of a substance across a membrane, the result os more or less the same, but the D depends on the membrane as well, and $\frac{\Delta\phi}{\Delta x}$ depends on the thickness of the membrane and the difference in concentration between the two sides
- if you look at it, it is identical to the Fourier law of heat conduction
- this is not a coincidence, the underlying model for the two is exactly the same
- this is what drives the diffusion of a gas into a liquid

Fick's second law

- gien the flux obtained with the first law, how does the overall concentration changes over time?
- point by point there is going to be influx from the regions with higher concetration, and efflux from the regions with higher concentration.
- the change over time is thus equal to the difference between the influx and the efflux
- both of these are driven by the first derivative of the concentration, so the change over time goes as the second derivative

$$\frac{\partial \phi}{\partial t} = D \frac{\partial \phi}{\partial x^2}$$

Atomic Interpretation

- Diffusion can be explained microscopically considering the motion of a big particle immersed in a medium composed of tiny particles
- the medium's particles are moving randomly due to their thermal energy, and doing so they hit the object in random directions
- each hit pushes the big molecule in a random direction, but the motion is rapidly lost due to the medium viscosity
- this is the origin of the brownian motion, observed by Brown in 1827, using a microscope to study the motion of pollen in water
- this phenomenon was theoretically explained by Einstein in 1905
- it was considered one of the first real evidences of the atomic theory

Stokes-Einstein equation

- if we consider a simple spherical object moving in a viscous medium, we can obtain a formal expression for the value of D based on the
    - Boltzmann's constant K_b
    - temperature of the medium T
    - radius of the object r
    - medium viscosity η
- this comes from a balance between the particles kinetical energy due to thermal motion and the stokes force acting on a sphere moving in a fluid

$$D = \frac{k_b T}{6\pi\eta r}$$

Einstein's relationship

- If we consider the motion of single atom following the diffusion's law, we can expect it to walk back and forth
- we could ask how much distance from the starting point it will have made after a certain amount of time
- the solution was obtained by Einstein, and was that the average distance from the origin x at time t in a medium with diffusion coefficient D will depend from the square root of the time
$$\bar{x}^2 = 2Dt \rightarrow \bar{x} = \sqrt{2Dt}$$
- this means that the diffusion "slows down", and while it quickly covers short distances, it takes long time to cover longer ones

Capillaries

- this puts a specific limitation on the structure that capillaries can have in the tissue.
- the cell consumption of oxygen limits how far can the oxygen diffuse in the tissue
- too many, and the tissue will be over oxygenated
- too little, and the cells will not be able to get oxygen at all
- different tissues will need to be perfused by a number of capillaries proportional to the average energy consumption of the tissue (ignoring other metabolic reasons, such as in the kidneys)

Partial pressure

Partial pressures - Henry's law

- Henry's law states that the concentration of gas dissolved in a liquid is proportional to the external pressure of that gas ouside of the liquid itself
- The dissolved gas will interact and react according to it's partial pressure
- the proportionality constant is called the equilibrium constant for the solvation process, and depends for the pair gas-solvant

equilibrium constant for the solvation

- It can be expressed in various forms depending on which uni of measurements one wants to use.
- the most commonly used (H^{cp}) is defined in terms of the concentrations c (numbers of moles per cubic meter) over the external pressure p

$$c = H^{cp} \cdot p$$

- O_2 in water: $H^{cp} = 1.3 \cdot 10{-}5\, mol/(m^3 \cdot Pa)$
- CO_2 in water: $H^{cp} = 3.4 \cdot 10{-}4\, mol/(m^3 \cdot Pa)$

vapor pressure

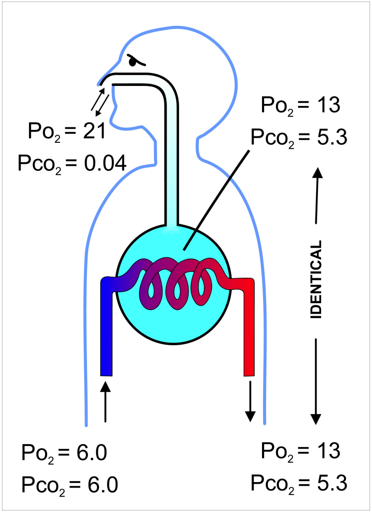
- At the surface of a liquid, the hard sphere collisions are what makes possibile evaporation at room temperature
- if an unusually hard hit happens nearby the surface, it can free a liquid molecule into the environment as a gas form (sublimation)
- at the same time, molecules of the medium in gas form can bind again to the surface and perform condensation
- over time, this results in a balance between the amount of molecules in liquid state and in gaseous state
- the resulting gas will have a certain pressure (determined by the environment temperature).
- this is called the vapor pressure of the liquid.
- can be also described an entropy driven phenomenon, the result is identical

vapor pressure and boiling point

- the temperature at which the vapor pressure is equal to the atmospheric pressure is the temperature at which the liquid starts to boil
- it is also the driving factor behind air embolism in divers: their blood contains gas disolved at high pressure, and if decompressed rapidly the vapor pressure gets higher than the vase pressure, causing the formation of bubbles
- this is why the standard therapy is recompression: by increasing the body pressure over the vapor pressure, the bubbles re-disolve, and then the pressure is reduced slowly to allow the body to get rid of the eccessive gas in the blood.

Respiration

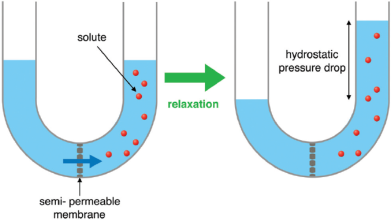
- Our body rely on diffusion to exchange oxygen and carbon dioxide with the environment
- in the time it takes for the air to enter and exit the lungs, the diffusion will quickly equalize the partial pressures (concentrations) of oxygen and carbon dioxide



pressures in KPa
from wikipedia

Osmosis

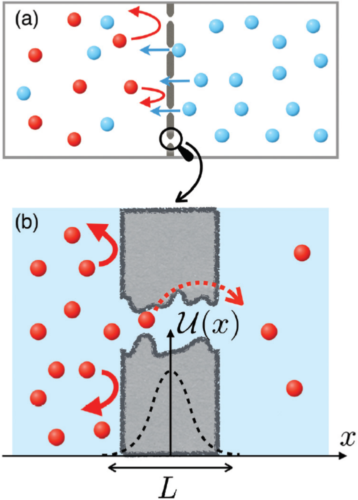
negative solute pressure Π



$$\Pi = K_\beta T \Delta C$$

where ΔC is the difference in concentration of the solute, T is the temperature in Kelvin, K_β is the Boltzmann constant.
It's not a form of diffusion, but of advection!
the process can be reversed, but requires an external pressure greater than Π

microscopical explaination



the solute molucelus bounce back on the barrier and push agains the water, creating a net force **away** from the membrane. it is a general process, does not depend on the type of solute of barrier.

Dialysis

- Dialysis is the artificial replica of the action normally performed by the kidneys
- it is a case of osmosis across a membrane, that has the property of letting through only molecules below a certain size
- this allow to separate the proteins and cells in the blood from other substances like urea
- there are also versions called **extra-corporeal CO2 removal**, where patients suffering from respiratory deficiencies (both chronic and acute) and have their blood's level of CO_2 restored, using a procedure equivalent to the traditional dialysis

Exercise

Flux in a capillar

- two big equal container with a volume of 20 liters filled with water
- one of the two has added glucose to a concentration of 0.2 molar
- the two containers are connected with a capillar, with a length of 50 cm and a diameter of 10 mm
- the diffusion coefficient of the glucose in water is $3 \cdot 10^{-6} cm^2/s$
- assuming that the concentration of glucose does not change significantly overall in the span of 25 hours, estimate:

1. the value of the flux J_s of glucose through the capillar
2. how many moles n_s moves overall in the 24 hours
3. is it correct to use the assumption of almost-constant concentration across the 24 hours

Flux in a capillar - solutions

- we can work all in centimeters, but need to convert moles per liters in moles per cubic centimeter
- $0.2mol/l \Rightarrow 2 \cdot 10 - 4mol/cm^3$
- the difference of concentration between the two containers is then $-2 \cdot 10 - 4mol/cm^3$
- we can use the Fick's equation as there is no motion of fluid
- we first obtain the flux per unit of area J_s
- then the total moles of glucose moves is J_s multiplied by the surface area and the time ellapsed (this is possible because the the assumption of constant concentration)

1. $J_s = -(3 \cdot 10^{-6} cm^2/s) \cdot (-2 \cdot 10^{-4} mol/cm^3)/(50cm) = 1.2 \cdot 10^{-11} mol/cm^2 s$
2. $n_s = J_S \cdot ((0.5cm)^2 \pi) \cdot (3600 \cdot 24) = 8.1 \cdot 10^{-7} mol$
3. $n_s \ll 0.2mol/l \cdot 20l \Rightarrow$yes