Waves and Oscillations

Enrico Giampieri

DIMES

Waves

waves

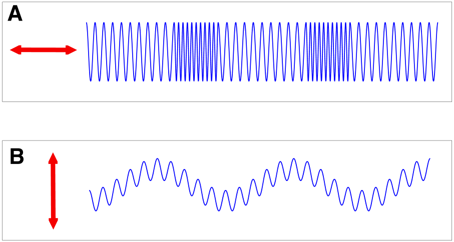
- A wave is a perturbation that propagates in space and time, sometimes (but not always) with characteristics of regularity and periodicity.
- Wave propagation: energy transmission without mass transport.
- the perturbation (mechanical, seismic, electromagnetic) is first confined to a zone of space, and progressively extends to other areas.
- the waves we are going to describe are small perturbations over the position of equilibrium, and they share a lot of characteristics with the harmonic motion
- the propagation comes from the interaction between neighboring elements of the medium

Dimensionality of the medium

- waves are often characterized based on the number of dimensions over which they can propagate
- 1-D wave: the end of a rope is disturbed, as in the whip, and this moves to the opposite end of the string.
- 2-D waves: stone in a pool of water, waves propagate on the surface as concentric circles of increasing radius starting from the point of impact of the stone.
- 3-D waves: sound waves. The air vibrates, for example by the vibration of a tuning fork, and this vibration propagates in space.

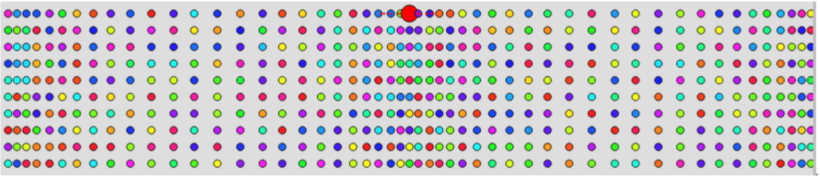
Wave types: propagation

- **transverse**: perturbation is orthogonal to propagation direction (sea waves, e.m. waves, "ola")
- **longitudinal** when the perturbation is along the propagation direction (sound waves)

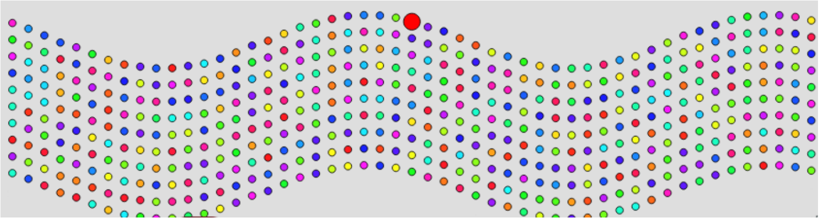


(A) Longitudinal wave.
(B) Transverse wave.
The red arrow indicates the direction of the external perturbation.

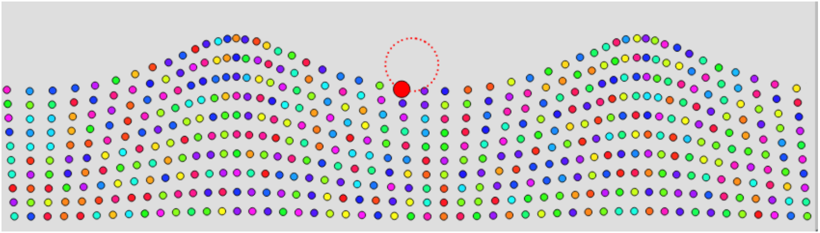
Longitudinal waves



Trasversal wave



Surface waves



Harmonic Waves

Wave parameters

- If the perturbation produces a simple (sinusoidal) periodic vibration, the generated wave is called a *harmonic* wave.
- The function that describes the propagation of a harmonic wave (**monochromatic plane wave**) has the following form:

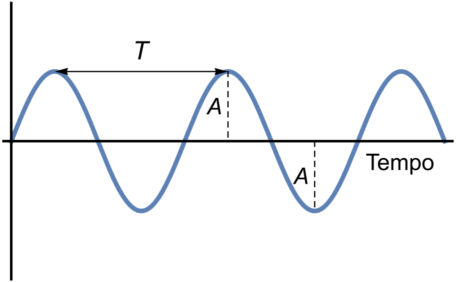
$$f(x,t) = A\sin(\omega t - kx + \phi) = A\sin\left(\frac{2\pi}{T}t - \frac{2\pi}{\lambda}x + \phi\right)$$

Wave description

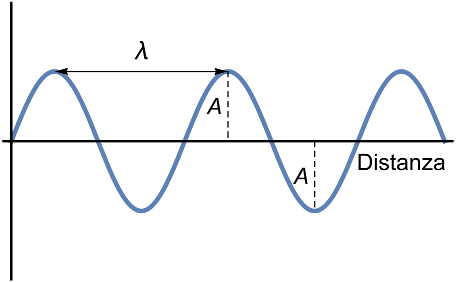
- Periodic function, similar to the equation for harmonic motion, but:
- The wave consists in the propagation of a physical quantity in time (pressure, electric field, height of a rope, etc), so the physical dimension of A depends on the type of wave
- The wave is a doubly periodic function, both in time and in space.

Double periodicity

- given a point of the wave in space we observe the periodic variation in time
- given a moment in time we observe the periodic form of the wave in space



Plot of the wave as a function of position.



Plot of the wave as a function of time.

period and frequency

- The "temporal" properties of the wave are identical to those encountered in the harmonic motions
- they are linked to the harmonic vibration of the source that generates the wave.
- The period T is the time for a point to perform a complete oscillation.
- The period T is related to the frequency ν:

$$\nu = \frac{1}{T}$$

period and frequency

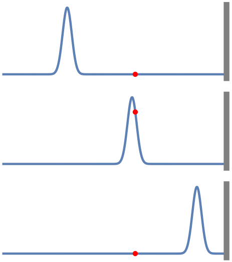
- In the I.S. the period is measured in seconds while the frequency, defined as the number of complete oscillations in the time unit, is measured in hertz (Hz).
- The wave characteristic quantities, wavelength λ and period T, are linked by a simple relation to the propagation velocity of the wave v, which is a constant feature:

$$v = \frac{\lambda}{T} = \lambda \cdot \nu$$

Wave parameters

- Amplitude A: maximum deviation from the equilibrium position, can be either positive or negative
- wavelength λ: distance between two maxima (or minima). Distance traveled by the wave during a complete oscillation (measured in meters)
- "wave number" $k = 2\pi/\lambda$ (analog of the frequency). Measured in m^{-1}: $\lambda = 2\pi/k$
- pulsation ω related to the wave oscillation frequency: $\omega = 2\pi\nu = 2\pi/T$,
- the phase ϕ: angle that represents the initial position of the wave with respect to the oscillation

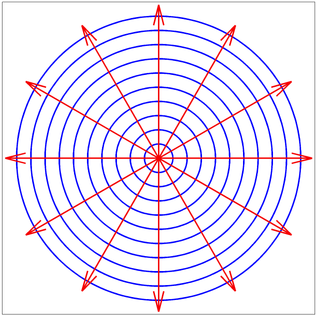
1-D wave on a rope

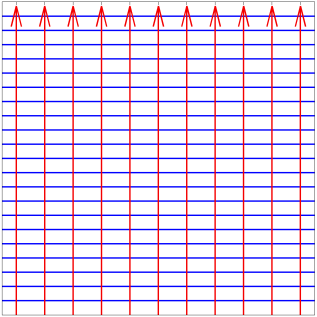


- The impulse propagates along the rope with a speed that depends on the nature of the rope and the applied force.
- The propagation speed also depends on the inertia and therefore on the mass of the rope.
- If we attach a mass to the rope at some point, when this point is reached by the perturbation, the mass gains a certain acceleration

$$v = \sqrt{\frac{F}{\rho A}}$$

Wave fronts: plane and spherical waves





- Wave fronts (blue lines): points where the wave has the same amplitude.
- Rays (red arrows): wave propagation directions.
- For the plane wave the directions are all parallel to each other, indicating that there is only one propagation direction.

Energy of a wave

- The total energy carried by the wave is given by the sum of the kinetic energy and the potential energy,
- we can calculate it for example when the material point is moved to the maximum distance from the equilibrium position
- this corresponding to the amplitude A, position where it is stopped (for which $v = 0$):

$$E_T = K + U = \frac{1}{2} m\omega^2 A^2$$

Energy of a wave

- Moreover, remembering the definition of average over a period, and calculating the average of kinetic energy and potential energy over a period of T
- we have that $< K >=< U >= 1/2E_T$, that is, the average kinetic energy is equal to the average potential energy and is worth half of the total energy
- (the <> symbol indicates the average, in our case over a time interval of T)

Intensity

- Intensity **I**, defined as the energy E that flows in the time unit t through a unit area S perpendicular to the wave propagation direction:

$$I = \frac{E}{S_{\perp} t}$$

- The unit area depends on the wave size.
- In the case of one-dimensional waves such as mechanical waves in a string, there is no unit area and the intensity is simply the energy carried through a point of the rope in the time unit.
- In the case of two-dimensional waves, such as waves in a water surface, the unit area is reduced to a segment of unit length.
- Only in the case of three-dimensional waves can one speak properly of a unitary area.

Intensity

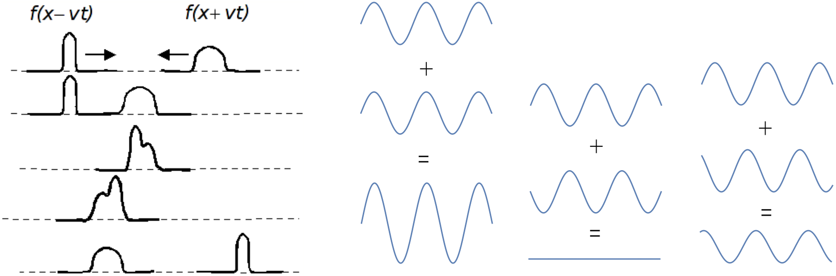
- In the case of a plane harmonic wave the intensity of the wave varies over time, and the mean intensity value $< I >$ over a period T is proportional to the square of the amplitude A of the wave
- this remains constant over time regardless of the distance from the source:

$$< I > \propto A^2$$

Superposition

Superposition principle

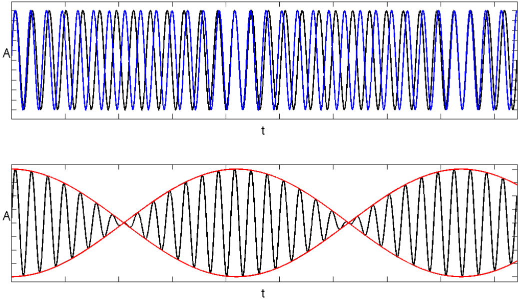
if two waves propagate in a given medium independently of each other, the overall perturbation is the sum of the perturbations due to each wave.



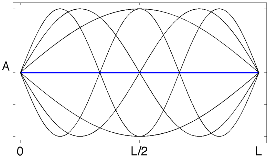
The amplitude of the resulting wave can be greater, less or equal to the sum of the initial amplitudes and their sum depends on the phase.

waves beats

- Overlap of two waves with a small difference in frequency.
- The waves will result in concordance and phase opposition periodically so as to have what is called temporal interference.
- If, for example, you vibrate two tuning forks of slightly different frequencies, you will hear a periodic sound of varying intensity: it is the phenomenon of beats.



Rope fixed at both ends



- The rope contains simultaneously both incident and reflected wave.
- The two waves therefore interfere.
- Stationary waves can have only specific frequencies (**harmonics**):

$$\nu_n = v\frac{n}{2L}\ ;\ (n = 1, 2, \ldots) \tag{1}$$

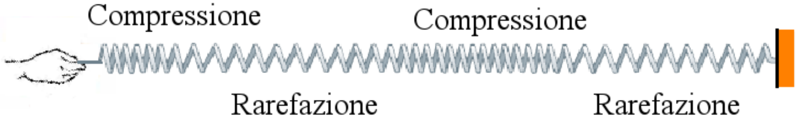
- where L is the length of the string and v the propagation velocity of the wave.

Fourier's theorem

- Any waves can be written as the sum of sinusoidal waves
- given the principle of superposition, we can describe the behavior of each one independently
- we can study each wave by itself, then recompose the final results
- once we know how to describe pure sinusoidal waves, we can describe the evolution of any wave

sound waves

- Sound waves (unlike mechanical waves in a string) are longitudinal waves, in which the vibration of the particles in the medium is in the same direction of propagation of the wave.
- For example, a series of compressions and expansions or rarefactions propagate along the spring.



- The sound waves, to propagate, require the existence of a medium whose particles are put into vibration.
- Usually this medium is the air, but sound can spread also in other media (water, solid) with different speed.

Material	*Velocity (m/s)*
Aria	343
Helium	1005
Hydrogen	1300
Water	1440
Wood	4000
Iron	5000

Sound waves are characterized by three parameters:

- Pitch or tone
- Intensity or volume
- Timbre or quality

Pitch

- It is part of everyone's experience to distinguish between acute and low sounds.
- Galileo first associated these sensations to the **frequency** of the sound wave: the smaller the **frequency**, the lower the sound, and viceversa.
- The human ear is able to perceive sounds which frequency is in the range $20 - 20000Hz$.

Infrasounds and Ultrasounds

- Sounds at frequencies greater than 20000 Hz are called ultrasounds and can be perceived by many animals.
- For example, dogs hear sounds of up to $50KHz$ and bats up to $100KHz$.
- Sounds less than $20Hz$ are called infrasounds.
- The earthquakes, the thunders and the vibrations of some heavy machinery are sources of infrasound.
- These sounds, although not audible, can affect the human body through resonance phenomena

Intensity

- The volume of a sound depends on the intensity of the wave.
- The intensity is proportional to the square of the wave amplitude, and represents the energy carried in the time unit through a unitary area perpendicular to the propagation direction(W/m^2).
- The human ear can perceive sounds of intensity between $10^{-12}W/m^2$ and $1W/m^2$; above this value there is a painful sensation.

Intensity perception

- The subjective perception of the **volume** of a sound is not proportional to the intensity of the sound wave.
- For example, a wave of intensity $10^{-2} W/m^2$ gives the sensation of a sound about twice as strong as a wave of intensity ten times smaller, that is $10^{-3} W/m^2$, and is four times stronger than an intensity wave $10^{-4} W/m^2$.
- Because of this nonlinear relationship between perceived volume and wave intensity, the intensity of the sound is measured through a **logarithmic scale**

Intensity measurement

- The unit of measure of audible sounds is *bel* (B), even if the *decibel* $(dB = 1/10B)$ is commonly used.
- The intensity level B of a wave is defined by the equation:

$$dB = 10 \cdot \log_{10}\left(\frac{I}{I_0}\right)$$

- where I is the intensity of the wave and I_0 is the reference intensity, usually equal to the threshold of audibility $(1.0 \times 10^{-12}\ W/m^2)$.

Intensity measurement

- For example, the intensity level of a sound whose intensity is $I = 1.0 \times 10^{-10} W/m^2$ will be

$$B = 10 \log_{10} \left(\frac{10^{-10}}{10^{-12}} \right) = 10 \log_{10} \left(10^2 \right) = 20 dB$$

- Note that the intensity level corresponding to the audibility threshold $10^{-12} W/m^2$ is $B = 10 \log_{10} \left(\frac{10^{-12}}{10^{-12}} \right) = 10 \log_{10} (1) = 0.$

Source	Volume (dB)	Intensity (W/m^2)
Rustle of leaves	10	1.0×10^{-11}
Whisper	20	1.0×10^{-10}
Conversation	65	3.2×10^{-6}
Traffic	70	1.0×10^{-5}
Hooter at 30 m	100	1.0×10^{-2}
Rock concert	120	1.0
Jet at 30 m	140	100

Table: Intensity of typical sounds

When we approach a shell to the ear and "hear the sea", we are actually amplifying the random noise produced by the motion of the air molecules, which otherwise is below the threshold of audibility.

Timbre

- If a violin and a piano emit a note with the same pitch, you can still perceive a clear difference between the two sounds: this difference is expressed as **timbre** or sound quality.
- If a vibrating tuning fork emits only a single frequency, a musical instrument will normally issue sounds from the **overlay** of various waves with different amplitudes (and frequencies).
- Considering stringed instruments, the stationary waves correspond to defined values of frequencies: the lowest frequency takes the name of fundamental, the successive ones are first, second harmonic and so on.

Frequency spectrum

- The sound emitted by an instrument can therefore be characterized by the set of frequencies that compose it and their relative amplitude.
- This procedure takes the name of **harmonic or spectral analysis**, and the result is described in the so-called Frequency Spectrum, a graph that expresses the intensity according to the frequencies present.

Frequency spectrum

- In general, every wave (sound, light, ECG, EEG, X-ray or Gamma) is characterized by its frequency spectrum
- a mathematical procedure (Fourier analysis) allows a wave to be decomposed into its fundamental harmonic components (as well as a vector can be decomposed into the sum of basic vectors).
- From this decomposition it is possible to deduce properties of the wave such as:
  - the type of instrument that produced the sound
  - the type of material that produced the light wave
  - the state of health of the heart observed by ECG
  - the radioactive material that emitted radiation ionizing.

Ultrasounds

- Sound waves can be used to characterize properties of materials, to estimate distances and speeds, and also to produce images of components (organs and tissues in the case of ultrasound) that could not be seen directly (for example enclosed in a compartment).
- When a wave passes from one medium to another it undergoes two phenomena:
- refraction, which alters the wavelength (and the direction) of the wave
- reflection, whereby a part of the wave reverses the direction returning towards source.
- When a sound wave passes through the air (or water) and meets an obstacle such as a rock face (or a seabed) it is reflected backwards through the same medium from which it arrived.

echo and sonars

- Knowing the velocity of propagation in the medium, we can characterize the uniform rectilinear motion along a path equal to twice the distance between the source and the reflecting body.
- By measuring the time taken by the signal to go back, we can then measure the distance between the source and the obstacle:

$$\Delta s = \frac{v\Delta t}{2}$$

- This principle is applied in sonar (sound waves) and in radar (electromagnetic waves), and also forms the basis of ultrasound (sound waves) in the medical field.

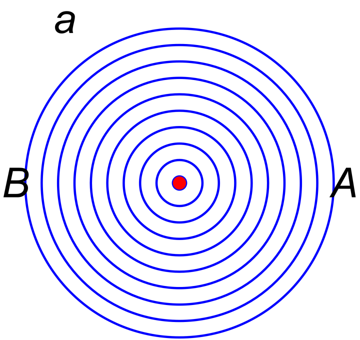
Ultrasound imaging

- Considering the body as a set of approximately homogeneous media (the interior of the various organs) separated by different "containers" (surfaces of the organs, membranes and interstitial tissues), a wave able to penetrate the body will be reflected by the various interfaces encountered.
- Given an instrument capable of generating sound waves with regularity and specific waveforms, and able to receive the signals returning from the various reflections and to measure their amplitude, based on the return time it is possible have an estimate of the reflected signal at a given depth.
- By collecting this information and integrating it with mathematical algorithms, it is possible to create an image that constitutes a sort of "shadow" of the organs crossed by the wave: two-dimensional *projections* of three-dimensional objects

- In a similar way to Computerized Tomography (CT, which uses different X-rays and properties of the interaction between waves and matter), ultrasounds capable of generating and collecting signals from different angles allow to reconstruct (starting from 2-D projections) a 3-D image of internal organs (or for example the fetus).
- Due to the characteristics of the media traversed within an organism, the sound waves used for this type of investigations are ultrasounds with frequencies of the order of MHz (typically $2 - 5MHz$)
- for waves of this frequency, in fact, the water (which makes up most of the tissues) is almost "transparent", so only the organ surfaces constitute potential reflecting surfaces.

Doppler effect

- if the source of the waves is in movement, one can observe variations in the wave properties
- the the source is immobile, the wave has the save wavelength in all directions
- when it's moving, the wavelengths get "compressed" in the direction of motion and "stretched" in the opposite direction



- the source is emitting waves with a frequency ν
- given v the speed of sound in the air $\lambda = \frac{v}{\nu}$ and $T = \frac{1}{\nu} = \frac{\lambda}{v}$
- if the source is getting closer to A with speed v_{SQR}, the difference in speed between the source and the sound wave is

$$v_R = v - v_{SQR}$$

- the wavelength corresponding to this (the one that an observer A in front of the source will measure) is given by

$$\lambda_R = v_R T$$

- this means that being $v_R < v$ the wavelength perceived by the observer A should be less than the original one

- in a similar way, if we consider an observer B located behind the source, that see it going away, the relative speed of the waves are going to be

$$v_R = v + v_{SQR}$$

- in this case, being $v_R > v$, the perceived wavelength perceived by the observer B is going to be greater than original one
- an example of this is the sound of an ambulance siren as it drives towards and the past us.
- while it drives toward us, the sound is perceived as a higher pitch
- while it drives away from us, the sound is perceived as a lower pitch

- the dopple effect can also be expressed in terms of changes of the perceived frequencies rather than perceived wavelengths
- let's see the proof for the approaching motion
    - A perceives a wavelength $\lambda_R = (v - v_{SOR})T = \lambda(1 - \frac{v_{SOR}}{v})$
    - the corresponding frequency is given by

$$\nu_R = \frac{v}{\lambda_R} = \frac{v}{\lambda(1 - \frac{v_{SOR}}{v})} = \frac{\nu}{1 - \frac{v_{SOR}}{v}}$$

    - given that $v_{SOR} < v$, the denominator is lower than 1 and the perceived frequency is greater than the emitted one $\nu_R > \nu$

- if we consider a source getting closer at a speed $v_{SOR} = 30m/s$
- it is emitting sound with frequency $\nu = 400\,Hz$
- an observer is going to perceive a sound with frequency ν_R given by

$$\lambda = \frac{400Hz}{1 - \frac{30m/s}{343m/s}} \cong 440Hz$$

- using the speed of sound in air $v = 343m/s$

- the Doppler effect can also show when is the observer to be in motion relative to the source
- given v_O the speed of the observer, using similar considerations as before to show that:
- for an observer that is going toward the source we have

$$\nu_R = \left(1 + \frac{v_O}{v}\right)\nu$$

- for an observer that is moving away from the source we have

$$\nu_R = \left(1 - \frac{v_O}{v}\right)\nu$$

- we can sinthetize all these relationships with a single expression

$$\nu_R = \left(\frac{v \pm v_O}{v \mp v_{SOR}}\right)\nu$$

- where the signs on top are relative of the case of source and observer getting closer, the lower one to the case of the source and observer moving away from each other

- when a sound wave is reflected by a moving object, the Doppler effect will happen twice:
- once because the object behaves as a incoming observer
- twice because it then emits the sounds waves as a moving source
- this effects are not symmetrical, but for objects moving slowly compared to the speed of sound they can be approximated as such
- for example an object moving at $4m/s$, hit by a way of frequency $6000\,Hz$
- as an observer, it will perceive a frequency of

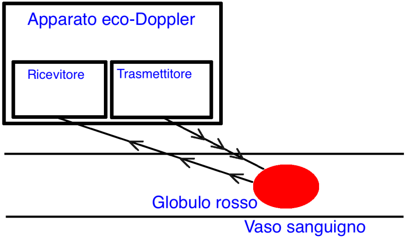
$$\nu' = \left(1 + \frac{v_O}{v}\right)\nu = \left(1 + \frac{4}{343}\right)6000 \cong 6070Hz$$

- an a sound source it will emit a frequency of

$$\nu'' = \frac{1}{1 - \frac{v_S}{v_O}}\nu' = \frac{1}{1 - \frac{4}{343}}6070 \cong 6140Hz$$

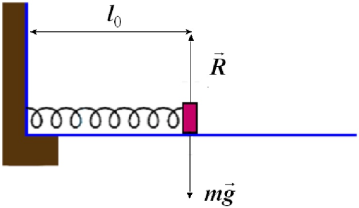
eco-Doppler fluximetry

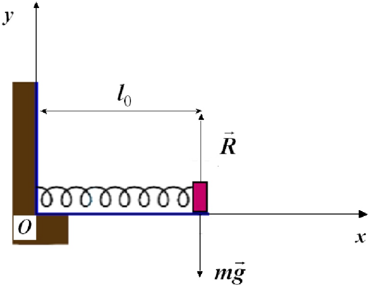
- is a technique that use ultrasounds (frequency of $5MHz$) on blood cells to estimate the speed and flux of blood in a non-invasive way
- the instrument is applied in contact with the skin, with a gel that avoid reflections due to pockets of air
- the emitter is also a receiver, that measure the change in frequency due to the reflection from the blood cells



resonance

Harmonic oscillator





- l_0 spring length at rest
- on y axis, weight and binding reaction compensate: $\vec{R} + m\vec{g} = 0$
- elastic force: $F_x = -k(x - l_0)$ k elastic constant
- if $x > l_0$, elongated spring $(F < 0)$, se $x < l_0$, compressed spring $(F > 0)$,$x = l_0$, $F = 0$.
- NOTE: setting x axis origin in l_0 I get $F_x = -kx$

Harmonic motion

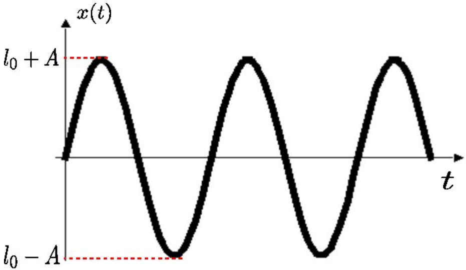
Given the elastic force, the solution is an harmonic (periodic) function:

$$x(t) = l_0 + A \cdot \sin(\omega t + \varphi) \rightarrow A \cdot \sin(\omega t + \varphi)$$

Motion confined to an interval of width $2A$ centered in $x = l_0$ (or in 0 changing origin).

$$l_0 - A \leq x(t) \leq l_0 + A \rightarrow -A \leq x(t) \leq A$$

Harmonic motion parameters



- **Period** $T = \frac{2\pi}{\omega}$
- **Frequency** $\nu == \frac{1}{T} \frac{\omega}{2\pi}$
- T time for a complete oscillation (of length 2A)
- frequency ν counts the number of oscillations in $1s$, and ω those in 2π seconds
- frequency and pulsation in S.I. are measured in s^{-1}, called Hertz (Hz).

Damped oscillations

If the oscillator moves in a viscous liquid, two forces act on the mass m: the elastic and friction, directed in opposite directions
$\vec{F}_{visc} = -\beta\vec{v} \quad \beta = 6\pi\eta r \quad \eta$ viscosity $\quad r$ radius.
in 1-D case $v_x = dx/dt$ thus force is $-\beta dx/dt$:

$$m\frac{d^2x}{dt^2} = -kx - \beta v = -kx - \beta\frac{dx}{dt}$$

In a simpler form:

$$\frac{d^2x}{dt^2} = -\omega^2(x - l_0) - C^2\frac{dx}{dt}$$

Setting $\omega = \sqrt{\frac{k}{m}}$ and $C = \sqrt{\frac{\beta}{m}}$

Damped oscillations - solution

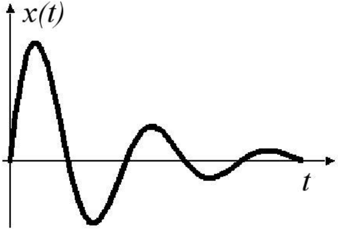
The solution is

$$x(t) = Ae^{-\frac{1}{2}C^2 t} \cdot \sin(\omega t + \varphi)$$

A and φ are two constants: amplitude and initial phase

$$A^{'} = Ae^{-\frac{1}{2}C^2 t} \Rightarrow x(t) = A^{'} \cdot \sin(\omega t + \varphi)$$

Amplitude A' **diminishes** in time with exponential decay.



NOTE: damped oscillator frequency ω is smaller than undamped ω_0.

Forced oscillations

To avoid damping we apply an external periodic force

$$F = F_0 \cdot \sin(\omega_{ext} t)$$

thus we obtain:

$$m\frac{d^2x}{dt^2} = -kx - \beta\frac{dx}{dt} + F_0 \cdot \sin(\omega_{ext} t)$$

Putting $\omega = \sqrt{\frac{k}{m}}$, $C = \sqrt{\frac{\beta}{m}}$ and $P = \frac{F_0}{m}$, equation becomes:

$$\frac{d^2x}{dt^2} = -\omega^2 x - C^2\frac{dx}{dt} + P \cdot \sin(\omega_{ext} t)$$

Forced oscillations - solution

Solution is still oscillatory:

$$x(t) = A' \cdot \sin(\omega_{ext} t + \delta)$$

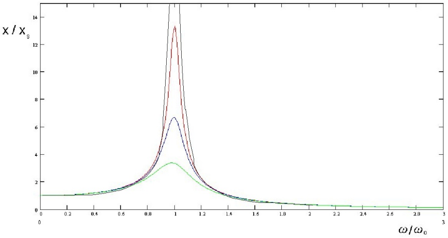
with

$$A = \frac{P}{\sqrt{(\omega_{ext}^2 - \omega^2)^2 + C^4 \omega_{ext}^2}}$$

and

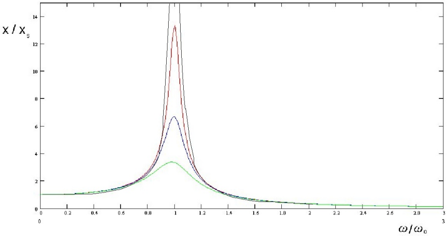
$$\delta = arctg \left[-\frac{C^2 \omega_{ext}}{\omega^2 - \omega_{ext}^2} \right]$$

Resonance



- The point oscillates with a frequency ω_{ext} equal to that of the external force applied
- while the amplitude A depends on the difference between the oscillation of the oscillator ω and that of the external force ω_{ext}.

Resonance

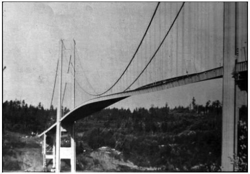


- When the forcing frequency ω_{ext} equals the natural frequency ω we have **resonance**
- it is the maximum **transfer of energy** from the external force to the oscillator

$$A = \frac{P}{C^2\omega} = \frac{F_0}{\beta\omega}$$

- the shape of the resonance curve depends on the amplitude of the damping coefficient β.

Examples



- The phenomenon of resonance is widespread in nature, and exploited in technological applications for centuries.
- In 1940, the Tacoma bridge collapsed due to the resonating oscillations triggered by the wind
- Resonance examples:
  - Clock with spring
  - Mechanical and acoustic resonance
  - Electric resonance (RLC circuits)
  - Optical resonance
  - Orbital and atomic resonance
  - Electronics and nuclear: NMR

Interpretation

- Each system (eg a solid) is characterized by one or more **proper frequencies** of oscillation (eg due to its mass and the type of material that composes it).
- An external forcing that has the **same** frequency of the system will send **maximum** energy to the oscillation of the system.
- There are therefore preferential modes (frequencies) of energy transmission through vibrations (electromagnetic waves, sound waves) based on the physical characteristics of the system.

Atomic resonance

- At the atomic level, photons carry energy as a function of the frequency of oscillation of e.m. field $E = h\nu$
- Many atomic-scale phenomena (electronic and nuclear excitation, atomic and molecular bonds) are characterized by specific ranges of energy E
- only photons with specific frequencies ν_{RIS} can transfer (yield) energy to atoms and produce such effects.

$$\nu_{RIS} = \frac{\Delta E}{h}$$

- Resonance governs many interactions between e.m. (but also sound) waves and matter, so the frequency that carries energy determines the type of phenomenon observed.

Exercise

sound waves energy

- a sound is emitted 200 m away from an observer with an intensity of $1\,W/m^2$ (measured 2 meter away from the source)
- the receiver's ear canal has a surface of approximately $2\,cm^2$
- what is the energy deposited in 10 seconds?

sound waves energy - solution

- the intensity of sound wave spread with and inverse square law
- the ration between the distances of measurements is $\frac{2}{200} = 100$
- this means that the intensity perceived is $\frac{1}{100^2} = 10^{-4}$ times weaker
- the receiver surface is $10^{-4} m^2$
- the power is thus $W = \frac{1\, W/m^2}{100^2} \cdot 10^{-4} m^2 = 10^{-8} W$
- in 10 seconds this means that the total energy deposited is $L = W \cdot \Delta t = 10^{-7} J$